Chapter 1

Double Integrals over Rectangular Regions

fixme change intro to reflect shorter chapter In this chapter, we extend this powerful idea into higher dimensions using the tools of multiple integration. While single integration enables us to calculate the area under a curve or the volume under a surface, multiple integration allows us to calculate volumes in three dimensions, and even hypervolumes in higher dimensions.

We start by discussing double integration, which allows us to find the volume under a surface in three dimensions. This method involves slicing the solid into infinitesimally small columns, and summing the volumes of these columns.

Next, we'll cover triple integration, a tool that lets us find the volume of more complicated solids in three-dimensional space. The idea is similar to double integration.

To properly implement these techniques, we'll also discuss the different coordinate systems that can be used in multiple integration, such as rectangular, cylindrical, and spherical coordinates, and when it's advantageous to use one system over another.

By the end of this chapter, you will have a deeper understanding of the techniques of multiple integration and how to apply them to find the volumes of various types of solids. The methods we study here will serve as a foundation for many topics in higher mathematics and physics, including electromagnetism, fluid dynamics, and quantum mechanics.

1.1 Double Integrals

Double integrals extend single-variable integration to functions of two variables, allowing us to calculate quantities like area, volume, and mass over a two-dimensional region. By integrating a function across a specified domain in the xy-plane, they help analyze how a quantity changes in both dimensions. Common in physics, engineering, and economics, double integrals involve setting up limits for the region and performing two successive integrations, often tailored to the region's geometry. We begin by discussing double integrals over rectangular regions, then extending that discussion to regions of any general shape. Finally, we discuss applications of double integrals.

1.1.1 Over Rectangular Regions

Suppose there is some function, $z = f(x, y)$, that is defined over the rectangular region, R, defined by $R = [a, b] \times [c, d] = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, and f is such that $f \geq 0$ for all $(x, y) \in \mathbb{R}$. Then the graph of f is a surface that lies above the rectangular region, R (see figure 1.1).

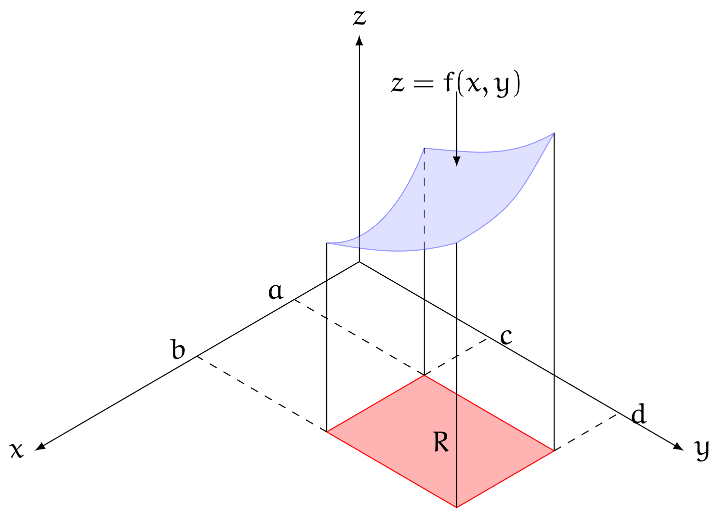

Figure 1.1: The graph of f over the region R

Let us call the solid that fills the space between the xy-plane and the surface $z = f(x, y)$ S. Formally, this is written as

$$S = \{(x, y, z) \in \mathbb{R}^3 | 0 \leq z \leq f(x, y), (x, y) \in \mathbb{R}\}$$

How can we find the volume of the solid, S? We will apply what we learned about Riemann sums and definite integrals in two dimensions to this three dimensional problem.

First, we divide R into rectangular subregions. We do this by dividing the interval $[a, b]$ into m subintervals with width $\Delta x = (b - a)/m$ and the interval $[c, d]$ into n subintervals with width $\Delta y = (d - c)/n$. Drawing lines through these divisions parallel to the x- and y-axes, we create a field of subrectangles, each with area $\Delta A = \Delta x \Delta y$ (see figure 1.2). Each subrectangle is defined by:

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] - \{(x, y)|x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

Since $f(x, y)$ in continuous over the R, there is some point, (x^*_{ij}, y^*_{ij}), equal to the average value of $f(x, y)$ over the subrectangle. Then we can approximate the volume between the xy-plane and $z = f(x, y)$ over the subrectangle as a column with base area ΔA and height $f(x^*_{ij}, y^*_{ij})$ (seefigure 1.3) and the volume of the column is given by:

$$V_{ij} = f(x^*_{ij}, y^*_{ij})\Delta A$$

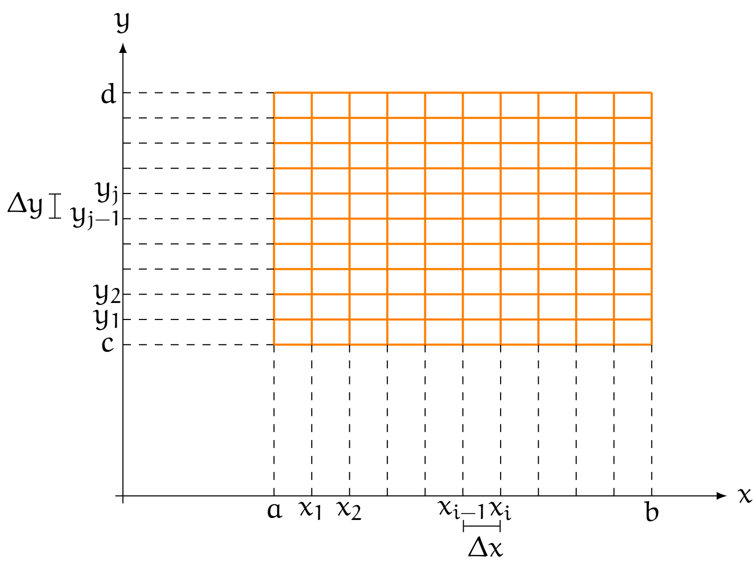

Figure 1.2: The region, R, on the xy-plane divided into subrectangles

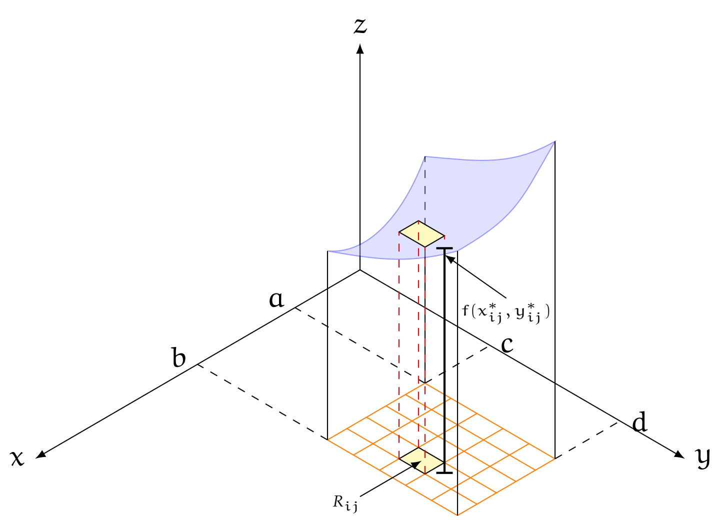

Figure 1.3: A single column with base ΔA and height $f(x_{ij}^*, y_{ij}^*)$

And therefore the approximate volume of the solid, S, that lies between the region, R, and $z = f(x, y)$ is the sum of all the columns over i and j:

$$V_S \approx \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

And just like with the area under a curve, we get the true volume by taking the limit as $n \to \infty$, which becomes a **double integral**:

Volume of a Solid over a Region

$$V_S = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R f(x, y)\, dA$$

1.2 Iterated Integrals

To be able to evaluate the double integral as outlined above, we must first discuss iterated integrals. Iterated integrals happen when you evaluate two single integrals, one inside the other. Consider some function, $g(x, y)$. We could integrate that function from $x = q$ to $x = r$ thusly:

$$\int_q^r g(x, y)\, dx$$

Notice that we are integrating with respect to x, so y terms will be treated as constants (recall partial differentiation: this is the opposite process). Let's call the result of this first integral $A(y)$:

$$A(y) = \int_q^r g(x, y)\, dx$$

We can then integrate the resulting function, $A(y)$, from $y = s$ to $y = t$:

$$\int_s^t A(y)\, dy = \int_s^t \left[\int_q^r g(x, y)\, dx \right]\, dy$$

This is called an **iterated integral**. When evaluating iterated integrals, we work from the inside out. You can also write it without the brackets:

$$\int_s^t \int_q^r g(x, y)\, dx\, dy$$

Example: evaluate the iterated integral $\int_0^3 \int_1^2 xy^2\, dy\, dx$.

Solution: We can re-write this to more explicitly show the inner and outer integrals:

$$\int_0^3 \left[\int_1^2 xy^2 \, dy\right] \, dx$$

As you can see, the inner integral is with respect to y. Let's isolate and evaluate the inner integral:

$$\int_1^2 xy^2 \, dy = x \int_1^2 y^2 \, dy = x \left[\frac{1}{3}y^2\right]_{y=1}^{y=2}$$

$$= \frac{x}{3}\left[2^3 - 1^3\right] = \frac{x}{3}[8-1] = \frac{7x}{3}$$

We were able to move x outside the integral because when we are integrating with respect to a specific variable (in this case, y), other variables are treated as constants. Now we can substitute $\int_1^2 xy^2 \, dy = \frac{7x}{3}$ into the iterated integral:

$$\int_0^3 \left[\int_1^2 xy^2 \, dy\right] \, dx = \int_0^3 \left[\frac{7x}{3}\right] \, dx$$

$$= \frac{7}{3}\left[\frac{1}{2}x^2\right]_{x=0}^{x=3} = \frac{7}{6}\left[3^2 - 0^2\right] = \frac{7 \cdot 9}{6} = \frac{21}{2}$$

Exercise 1 Order of Evaluating Iterated Integrals

Show that $\int_0^3 \int_1^2 xy^2 \, dy \, dx = \int_1^2 \int_0^3 xy^2 \, dx \, dy$.

Working Space

Answer on Page 9

Exercise 2 Evaluating Iterated Integrals

Evaluate the following iterated integrals.

1. $\int_0^1 \int_1^2 \left(x + e^{-y}\right)\, dx\, dy$
2. $\int_{-3}^3 \int_0^{\pi/2} \left(2y + y^2 \cos x\right)\, dx\, dy$
3. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3 \theta\, d\theta\, dt$

Working Space

Answer on Page 9

1.3 Fubini's Theorem for Double Integrals

Fubini's theorem states that for a function, f, that is continuous over the rectangular region, R, the double integral of f over the region $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$ is equal to the iterated integral of f with respect to x and y. This is expressed mathematically below:

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x, y)\,|a \le x \le b, c \le y \le d\}$, then

$$\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Exercise 3 Applying Fubini's Theorem

Working Space

Rewrite the following double integrals as iterated integrals.

1. $\iint_R \frac{xy^2}{x^2+1}\ dA,\ R = \{(x, y)|0 \le x \le 1, -3 \le y \le 3\}$
2. $\iint_R \frac{\sec\theta}{\sqrt{1+t^2}}\ dA,\ R = \{(\theta, t)|0 \le \theta \le \frac{\pi}{4}, 0 \le t \le 1\}$

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Exercise 4

Evaluate the double integral.

1. $\iint_R \frac{xy^2}{x^2+1}\, dA$, $R = \{(x, y) \mid 0 \leq x \leq 2, \ -3 \leq y \leq 3\}$
2. $\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\, dA$, $R = \{(\theta, t) \mid 0 \leq \theta \leq \pi/3, \ 0 \leq t \leq \frac{1}{2}\}$
3. $\iint_R x \sin(x+y)\, dA$, $R = [0, \pi/6] \times [0, \pi/3]$

Working Space

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Appendix A

Answers to Exercises

Answer to Exercise ?? (on page 5)

We have already shown that $\int_0^3 \int_1^2 xy^2 \, dy \, dx = \frac{21}{2}$. We will evaluate $\int_1^2 \int_0^3 xy^2 \, dx \, dy$ and see if we get the same result.

$$\int_0^3 xy^2 \, dx = y^2 \int_0^3 x \, dx = y^2 \left[\frac{1}{2}x^2\right]_{x=0}^{x=3}$$

$$= \frac{y^2}{2}\left[3^2 - 0^2\right] = \frac{9y^2}{2}$$

Substituting this back into the iterated integral:

$$\int_1^2 \int_0^3 xy^2 \, dx \, dy = \int_1^2 \frac{9y^2}{2} \, dy = \frac{9}{2}\int_1^2 y^2 \, dy$$

$$= \frac{9}{2}\left[\frac{1}{3}y^3\right]_{y=1}^{y=2} = \frac{9}{2} \cdot \frac{1}{3}\left[2^3 - 1^3\right]$$

$$= \frac{3}{2}(8-1) = \frac{21}{2}$$

Answer to Exercise 2 (on page 6)

1. Answer: $\frac{5}{2} - \frac{1}{e}$. Solution: $\int_0^1 \int_1^2 \left(x + e^{-y}\right) \, dx \, dy = \int_0^1 \left(\frac{1}{2}x^2 + xe^{-y}\right)\Big|_{x=1}^{x=2} dy = \int_0^1 \left(2 - \frac{1}{2} + 2e^{-y} - e^{-y}\right)$ d $\int_0^1 \left(\frac{3}{2} + e^{-y}\right) \, dy = \left[\frac{3}{2}y - e^{-y}\right]_{y=0}^{y=1} = \left(\frac{3}{2}(1) - e^{-1}\right) - \left(\frac{3}{2}(0) - e^0\right) = \frac{5}{2} - \frac{1}{e}$

2. Answer: 18. Solution: $\int_{-3}^3 \int_0^{\pi/2} \left(2y + y^2 \cos x\right) \, dx \, dy = \int_{-3}^3 \left[2xy + y^2 \sin x\right]_{x=0}^{x=\pi/2} \, dy = \int_{-3}^3 \left[\left(\pi y + y^2\right) - (0+0)\right] \, dy = \int_{-3}^3 \left(\pi y + y^2\right) \, dy = \left[\frac{\pi}{2}y^2 + \frac{1}{3}y^3\right]_{y=-3}^{y=3} = \left(\frac{\pi}{2}(9) + \frac{1}{3}(27)\right) - \left(\frac{\pi}{2}(9) + \frac{1}{3}(-27)\right) = 9 - (-9) = 18$

3. Answer: 6. Solution: $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\theta \, d\theta \, dt = \left(\int_0^3 t^2 \, dt\right) \times \left(\int_0^{\pi/2} \sin^3\theta \, d\theta\right) = \left[\frac{1}{3}t^3\right]_{t=0}^{t=3} \times \left(\int_0^{\pi/2} \sin\theta \sin^2\theta \, d\theta\right) = 9\int_0^{\pi/2} \sin\theta \left(1 - \cos^2\theta\right) \, d\theta = 9\left[\int_0^{\pi/2} \sin\theta \, d\theta - \int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta\right] = 9\left[(-\cos\theta)\Big|_{\theta=0}^{\theta=\pi/2} + \left(\frac{1}{3}\cos^3\theta\right)\Big|_{\theta=0}^{\theta=\pi/2}\right] = 9\left[-(-\cos 0) + (-\frac{1}{3}\cos^3 0)\right] = 9\left(1 - \frac{1}{3}\right) = 9\left(\frac{2}{3}\right) = 6$

Answer to Exercise 3 (on page 7)

1. $\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx$ OR $\int_{-3}^3 \int_0^1 \frac{xy^2}{x^2+1}\, dx\, dy$
2. $\int_0^{\pi/4} \int_0^1 \frac{\sec\theta}{\sqrt{1+t^2}}\, dt\, d\theta$ OR $\int_0^1 \int_0^{\pi/4} \frac{\sec\theta}{\sqrt{1+t^2}}\, d\theta\, dt$

Answer to Exercise 4 (on page 8)

1. $\iint_R \frac{xy^2}{x^2+1}\, dA$, $R = \{(x,y)\,|0 \le x \le 2, -3 \le y \le 3\} = \int_0^2 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx = \int_0^2 \frac{x}{x^2+1}\, dx \cdot \int_{-3}^3 y^2\, dy$. To evaluate the integral with respect to x, we use the u-substitution $u = x^2+1$, $(x)dx = \frac{1}{2}du$: $\int_0^2 \frac{x}{x^2+1}\, dx \cdot \int_{-3}^3 y^2\, dy = \int_{x=0}^{x=2} \frac{1}{2}\frac{1}{u}\, du \cdot \int_{-3}^3 y^2\, dy = \frac{1}{2}\ln|u||_{x=0}^{x=2} \cdot \frac{1}{3}\left[y^3\right]_{y=-3}^{y=3} = \frac{1}{2}\left[\ln\left(2^2+1\right) - \ln\left(0^2+1\right)\right] \cdot \frac{1}{3}\left[3^3 - (-3)^3\right] = \frac{1}{2}\ln 5 \cdot \frac{1}{3}\left(27 - (-27)\right) = \frac{\ln 5}{2}\frac{54}{3} = 9\ln 5$
2. $\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\, dA$, $R = \{(\theta,t)\,|\, 0 \le \theta \le \pi/3,\ 0 \le t \le \frac{1}{2}\} = \int_0^{\pi/3}\int_0^{1/2} \frac{\tan\theta}{\sqrt{1-t^2}}\, dt\, d\theta = \left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt\right]$. Recall that $\frac{d}{dt}\arcsin t = \frac{1}{\sqrt{1-t^2}}$. Applying FTC, then $\left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\, dt\right] = \left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\arcsin t\right]_{t=0}^{t=1/2} = \left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\arcsin\frac{1}{2} - \arcsin 0\right] = \left[\int_0^{\pi/3} \tan\theta\, d\theta\right] \cdot \left[\frac{\pi}{6}\right] = \frac{\pi}{6}\int_0^{\pi/3} \frac{\sin\theta}{\cos\theta}\, d\theta$. To evaluate this final integral, we use the u-substitution $u = \cos\theta$ and $-du = \sin\theta d\theta$: $\frac{\pi}{6}\int_0^{\pi/3} \frac{\sin\theta}{\cos\theta}\, d\theta = -\frac{\pi}{6}\int_{\theta=0}^{\theta=\pi/3} \frac{1}{u}\, du = -\frac{\pi}{6}\ln u|_{\theta=0}^{\theta=\pi/3} = -\frac{\pi}{6}\left[\ln(\cos\theta)\right]_{\theta=o}^{\theta=\pi/3} = \frac{\pi}{6}\left[\ln(\cos 0) - \ln\left(\cos\frac{\pi}{3}\right)\right] = \frac{\pi}{6}\left[\ln 1 - \ln\frac{1}{2}\right] = \frac{\pi}{6}\ln\frac{1}{1/2} = \frac{\pi}{6}\ln 2$
3. $\iint_R x\sin(x+y)\, dA$, $R = [0,\pi/6] \times [0,\pi/3] = \int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\, dy\, dx$. Recall the sum formula for sine:

   $$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

   We can substitute this into our iterated integral:

   $$\begin{aligned}\int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\, dy\, dx &= \int_0^{\pi/6}\int_0^{\pi/3} x\left[\sin x\cos y + \cos x\sin y\right]\, dy\, dx \\ &= \int_0^{\pi/6}\left[\int_0^{\pi/3} x\sin x\cos y\, dy + \int_0^{\pi/3} x\cos x\sin y\, dy\right]\, dx\end{aligned}$$

   Let us designate $\int_0^{\pi/3} x\sin x\cos y\, dy$ as integral **A** and $\int_0^{\pi/3} x\cos x\sin y\, dy$ as integral **B**. First, we will evaluate integral **A**:

   $$\int_0^{\pi/3} x\sin x\cos y\, dy = x\sin x\int_0^{\pi/3} \cos y\, dy$$

$$= x\sin x\,[\sin y]_{y=0}^{y=\pi/3} = x\sin x\left[\sin\frac{\pi}{3} - \sin 0\right]$$

$$= x\sin x\left(\frac{\sqrt{3}}{2}\right) = \frac{x\sqrt{3}}{2}\sin x$$

Next we evaluate integral **B**:

$$\int_0^{\pi/3} x\cos x\sin y\ dy = x\cos x\int_0^{\pi/3}\sin y\ dy$$

$$= x\cos x\,[-\cos y]_{y=0}^{y=\pi/3} = x\cos x\left[-\cos\frac{\pi}{3} - (-\cos 0)\right]$$

$$= x\cos x\left[-\frac{1}{2} - (-1)\right] = \frac{x}{2}\cos x$$

Substituting back in for integrals **A** and **B**:

$$\int_0^{\pi/6}\left[\int_0^{\pi/3} x\sin x\cos y\ dy + \int_0^{\pi/3} x\cos x\sin y\ dy\right]\ dx = \int_0^{\pi/6}\left[\frac{x\sqrt{3}}{2}\sin x + \frac{x}{2}\cos x\right]\ dx$$

$$= \frac{\sqrt{3}}{2}\int_0^{\pi/6} x\sin x\ dx + \frac{1}{2}\int_0^{\pi/6} x\cos x\ dx$$

Again, we will designate $\int_0^{\pi/6} x\sin x\ dx$ as integral **C** and $\int_0^{\pi/6} x\cos x\ dx$ as integral **D**. We start by using integration by parts to evaluate integral **C**:

Let $u = x$ and $dv = \sin x dx$. Then $v = -cos x$ and $du = dx$ and therefore:

$$\int_0^{\pi/6} x\sin x\ dx = [x(-\cos x)]_{x=0}^{x=\pi/6} - \int_0^{\pi/6}(-\cos x)\ dx$$

$$= \left[\frac{\pi}{6}\left(-\cos\frac{\pi}{6}\right)\right] - [0(-\cos 0)] + \sin x|_{x=0}^{x=\pi/6}$$

$$= -\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2} - 0 + \sin\frac{\pi}{6} - \sin 0 = \frac{1}{2} - \frac{\pi\sqrt{3}}{12} = \frac{6-\pi\sqrt{3}}{12}$$

Next, we will use integration by parts to evaluate integral **D**. Let $u = x$ and $dv = \cos x dx$. Then $du = dx$ and $v = \sin x$ and therefore:

$$\int_0^{\pi/6} x\cos x\ dx = [x\sin x]_{x=0}^{x=\pi/6} - \int_0^{\pi/6}\sin x\ dx$$

$$= \left[\frac{\pi}{6}\sin\frac{\pi}{6} - 0\sin 0\right] - (-\cos x)\,|_{x=0}^{x=\pi/6} = \frac{\pi}{6}\cdot\frac{1}{2} + \cos\frac{\pi}{6} - \cos 0$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

Substituting back in for integrals **C** and **D**:

$$\frac{\sqrt{3}}{2}\int_0^{\pi/6} x\sin x\ dx + \frac{1}{2}\int_0^{\pi/6} x\cos x\ dx = \frac{\sqrt{3}}{2}\left(\frac{6-\pi\sqrt{3}}{12}\right) + \frac{1}{2}\left(\frac{\pi+6\sqrt{3}-12}{12}\right)$$

$$= \frac{6\sqrt{3} - 3\pi + \pi + 6\sqrt{3} - 12}{24} = \frac{6\sqrt{3} - 6 - \pi}{12}$$

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