Chapter 1

Trigometric Functions

As mentioned earlier, in a right triangle where one angle is θ, the sine of θ is the length of the side opposite θ divided by the length of the hypotenuse.

The sine function is defined for any real number. We treat that real number θ as an angle, we draw a ray from the origin out to the unit circle. The y value of that point is the sine. So, for example, the $\sin(\frac{4\pi}{3})$ is $-\sqrt{3}/2$ (see figure 1.1).

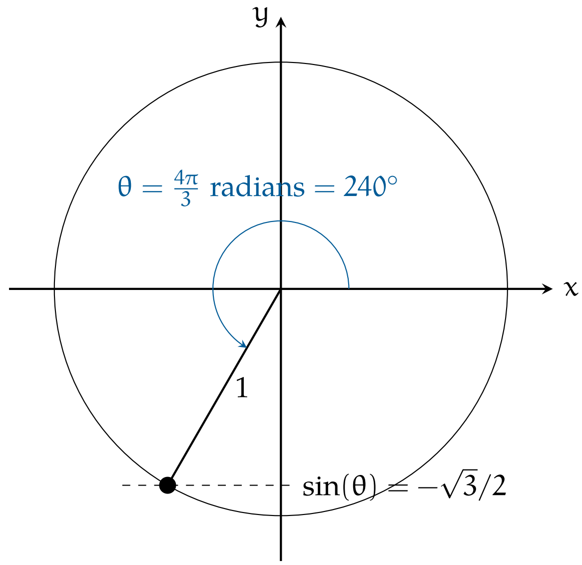


Figure 1.1: $\sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}$

(Note that in this section, we will be using radians instead of degrees unless otherwise noted. While degrees are more familiar to most people, engineers and mathematicians nearly always use radians when solving problems. Your calculator should have a radians mode and a degrees mode. You want to be in radians mode.)

Similarly, we define cosine using the unit circle: to find the cosine of θ, we draw a ray from the origin at the angle θ. The x component of the point where the ray intersects the unit circle is the cosine of θ (shown in figure 1.2).

From this description, it is easy to see why $\sin(\theta)^2 + \cos(\theta)^2 = 1$. They are the legs of a right triangle with a hypotenuse of length 1.

It should also be easy to see why $\sin(\theta) = \sin(\theta + 2\pi)$: Each time you go around the circle, you come back to where you started.

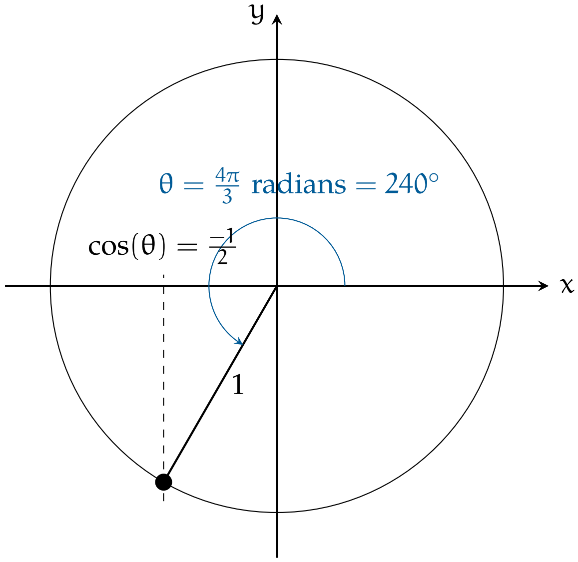
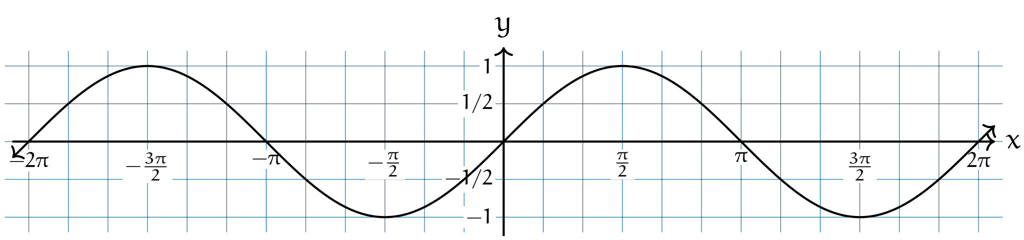


Figure 1.2: $\cos\frac{4\pi}{3} = \frac{-1}{2}$

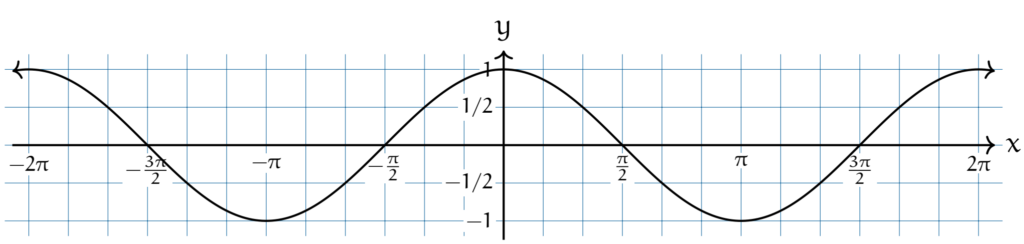
Can you see why $\cos(\theta) = \sin(\theta + \pi/2)$? Turn the picture sideways.

1.1 Graphs of sine and cosine

Here is a graph of $y = \sin(x)$:



It looks like waves, right? It goes forever to the left and right. Remembering that $\cos(\theta) = \sin(\theta + \pi/2)$, we can guess what the graph of $y = \cos(x)$ looks like:

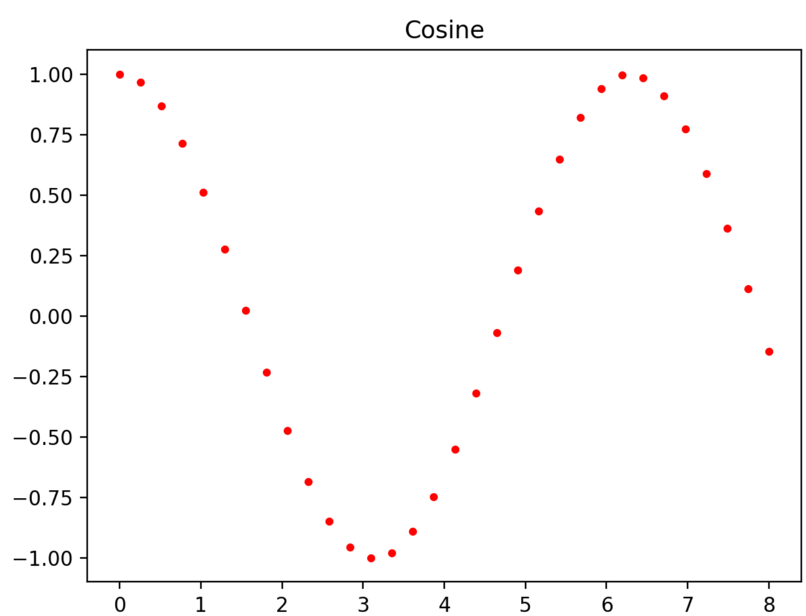


1.2 Plot cosine in Python

Create a file called `cos.py`:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(thetas, cosines, 'r.', label="Cosine")
ax.set_title("Cosine")
plt.show()
```

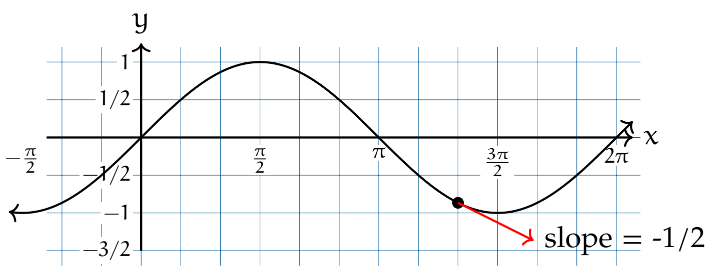
This will plot 32 points on the cosine wave between 0 and 8. When you run it, you should see something like this:



1.3 Derivatives of trigonometic functions

Here is a wonderful property of sine and cosine functions: At any point θ, the slope of the sine graph at θ equals $\cos(\theta)$.

For example, we know that $\sin(4\pi/3) = -(1/2)\sqrt{3}$ and $\cos(4\pi/3) = -1/2$. If we drew a line tangent to the sine curve at this point, it would have a slope of -1/2:



We say "The derivative of the sine function is the cosine function."

Can you guess the derivative of the cosine function? For any θ, the slope of the graph of the $\cos(\theta)$ is $-\sin(\theta)$.

Exercise 1 Derivatives of Trig Functions Practice 1

Use the limit definition of a derivative to show that $\frac{d}{dx}\cos x = -\sin x$

Working Space

Answer on Page 11

The derivatives of all the trigonometric functions are presented below:

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\sec x \sec x \cdot \tan x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$

Example: Find the derivative of $f(x)$ if $f(x) = x^2 \sin x$ **Solution**: Using the product rule, we find that:

$$\frac{d}{dx}f(x) = (x^2)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)$$

Taking the derivatives:

$$= x^2(\cos x) + 2x(\sin x)$$

Exercise 2 Derivatives of Trig Functions 2

Find the derivative of the following functions:

1. $f(x) = \frac{\sec x}{1+\tan x}$
2. $y = \sec t \tan t$
3. $f(\theta) = \frac{\theta}{4-\tan\theta}$
4. $f(t) = 2\sec t - \csc t$
5. $f(\theta) = \frac{\sin\theta}{1+\cos\theta}$
6. $f(x) = \sin x \cos x$

Working Space

Answer on Page 12

1.4 A weight on a spring

Let's say you fill a rollerskate with heavy rocks and attach it to the wall with a stiff spring. If you push the skate toward the wall a release it, it will roll back and forth. Engineers would say "The skate will oscillate."

Intuitively, you can probably guess:

- If the spring is stronger, the skate will oscillate more times per minute.
- If the rocks are lighter, the skate will oscillate more times per minute.

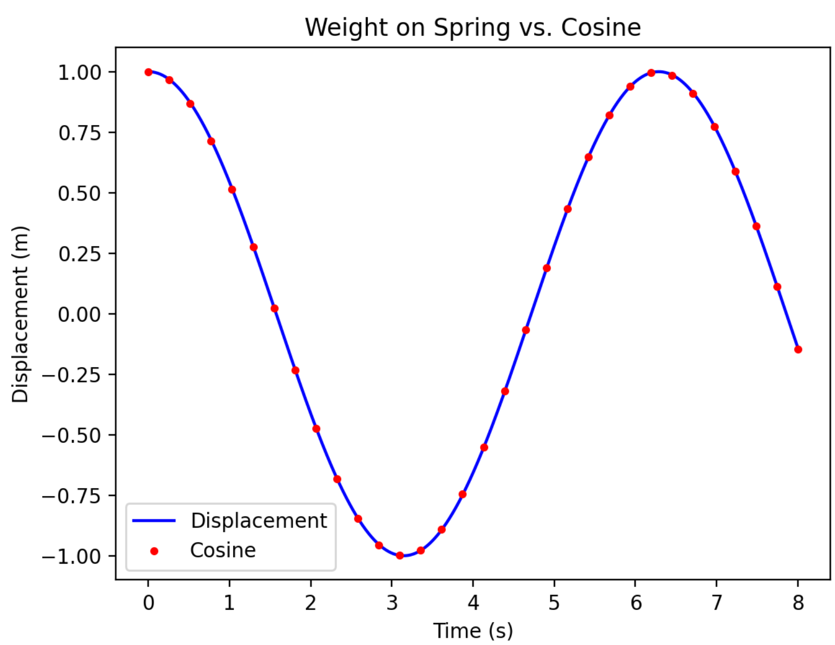
The force that the spring exerts on the skate is proportional to how far its length is from its relaxed length. When you buy a spring, the manufacturer advertises its "spring rate", which is in pounds per inch or newtons per meter. If a spring has a rate of 5 newtons per meter, which means that if stretch or compress it 10 cm, it will push back with a force of 0.5 newtons. If you stretch or compress it 20 cm, it will push back with a force of 1 newton.

Let's write a simulation of the skate-on-a-spring. Duplicate cos.py, and name the new copy spring.py. Add code to implement the simulation:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Constants
mass = 100 # kg
spring_constant = -1 # newtons per meter displacement
time_step = 0.01 # s

# Initial state
displacement = 1.0 # height above equilibrium in meters
velocity = 0.0
time = 0.0 # seconds

# Lists to gather data
displacements = []
times = []

# Run it for a little while
while time <= until:
    # Record data
    displacements.append(displacement)
    times.append(time)

    # Calculate the next state
    time += time_step
    displacement += time_step * velocity
    force = spring_constant * displacement
    acceleration = force / mass
    velocity += acceleration

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(times, displacements, 'b', label="Displacement")
ax.plot(thetas, cosines, 'r.', label="Cosine")
```

```
ax.set_title("Weight on Spring vs. Cosine")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Displacement (m)")
ax.legend()
plt.show()
```

When you run it, you should get a plot of your spring and the cosine graph on the same plot.



The position of the skate is following a cosine curve. Why?

Because a sine or cosine waves happen whenever the acceleration of an object is proportional to -1 times its displacement. Or in symbols:

$$a \propto -p$$

where a is acceleration and p is the displacement from equilibrum.

Remember that if you take the derivative of the displacement, you get the velocity. And if you take the derivative of that, you get acceleration. So, the weight on the spring must follow a function f such that

$$f(t) \propto -f''(t)$$

Remember that the derivative of the $\sin(\theta)$ is $\cos(\theta)$.

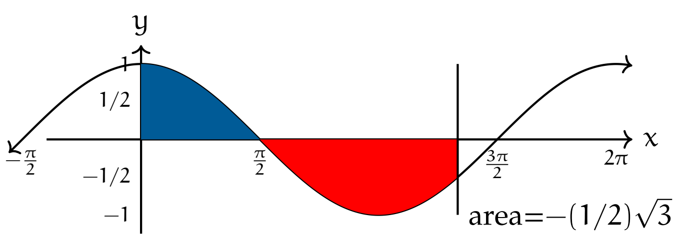
And the derivative of the $\cos(\theta)$ is $-\sin(\theta)$

Thus these sorts of waves have an almost-magical power: their acceleration is proportional to -1 times their displacement.

Thus sine waves of various magnitudes and frequencies are ubiquitous in nature and technology.

1.5 Integral of sine and cosine

If we take the area between the graph and the x axis of the cosine function (and if the function is below the x axis, it counts as negative area), from 0 to $4\pi/3$, we find that it is equal to $-(1/2)\sqrt{3}$



We say “The integral of the cosine function is the sine function.”

1.5.1 Integrals of Trig Functions Practice

Exercise 3

Evaluate the following integrals:

1. $\int \sec x \tan x \, dx$

Working Space

Answer on Page 12

This is a draft chapter from the Kontinua Project. Please see our website (https://kontinua.org/) for more details.

APPENDIX A

Answers to Exercises

Answer to Exercise 1 (on page 4)

We start by writing out the limit:

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos x + h - \cos x}{h}$$

Applying the sum formula for $\cos(x+h)$, we get:

$$= \lim_{h\to 0}\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

Rearranging to group the $\cos x$ and applying the Difference Rule:

$$= \lim_{h\to 0}\frac{\cos x \cos h - \cos x}{h} - \lim_{h\to 0}\frac{\sin x \sin h}{h}$$

Applying the Constant Multiple Rule:

$$= \cos x \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h}$$

Recalling that $\lim_{h\to 0}\frac{\sin h}{h} = 1$,

$$= \cos x \lim_{h\to a}\frac{\cos h - 1}{h} - \sin x \cdot 1$$

Recalling that $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$:

$$= \cos x \cdot 0 - \sin x = -\sin x$$

Therefore, $\frac{d}{dx}\cos x = -\sin x$

Answer to Exercise 2 (on page 5)

1. $\frac{\sec x(\tan x-1)}{(1+\tan x)^2}$
2. $\sec t[\sec^2 t + \tan^2 t]$
3. $\frac{4-\tan\theta+\theta\sec^2\theta}{(4-\tan\theta)^2}$
4. $2\sec t\tan t + \csc t\cot t$
5. $\frac{2}{(1+\cos\theta)^2}$
6. $\cos^2 x - \sin^2 x$

Answer to Exercise 3 (on page 8)

1. $\sec x + C$