Chapter 1

Matrices as Transformations

Recall that a *function*, informally, is a rule that takes an input and produces an output. For example, we know the common function $f(x) = x^3$ takes the input and cubes it, such that the result is $x \times x \times x$. Inputting $x = 3$ outputs $f(3) = 27$.

In Linear Algebra, we can think of matrices as a type of function. Recall our systems of matrices equation,

$$A\vec{x} = \vec{b}$$

We can rearrange this to look closer to function notation:

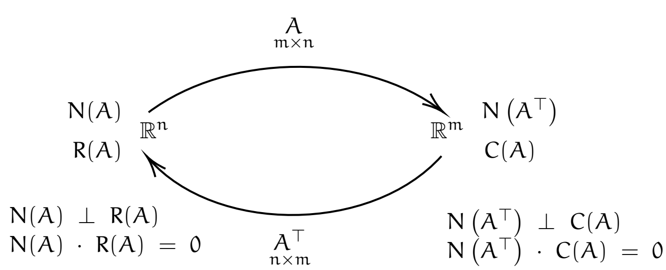
$$\vec{b} = A\vec{x}$$

We can change our input vector $\vec{x}$, which directly affects the output variable $\vec{b}$.

Recall our subspace diagram; the matrix A, which is size $m \times n$, takes the input vectors $\vec{x} \in \mathbb{R}^n$, and transforms them to the output $\vec{b} \in \mathbb{R}^m$.

The transformation T is said to map from the reals of n to the reals of m, such that: $T : \mathbb{R}^n \to \mathbb{R}^m$

- $\mathbb{R}^n$ is called the **domain**
- $\mathbb{R}^m$ is called the **codomain**



Let's call the transformation T, such that

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

We can rewrite our transformation with matrix A

$$T(\vec{x}) = A\vec{x}$$

A matrix transformation is completely determined by where it sends the *standard basis vectors*. Each column of A is the image of a basis vector under the transformation, and every output vector is a linear combination of these columns.

Consequently, the column space of A represents the full set of possible outputs of the transformation.

1.0.1 The Identity Transformation

For this chapter, we will restrict our transformations to $\mathbb{R}^2 \to \mathbb{R}^2$, such that our matrix is 2×2. This will help calculations be simpler and also will allow us to make easy to follow transformation diagrams.

Let's take the simplest transformation; the *identity transformation*. Take the matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Every vector in the subspace of $\mathbb{R}^n$ **stays where it is**.

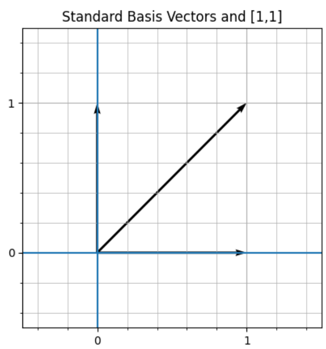
- e_1 stays e_1
- e_2 stays e_2

Every vector stays where it originally started in this transformation. In a way, no transformation is truly applied.

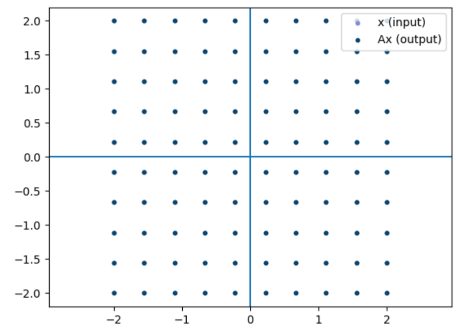
Under the identity transformation, nothing moves. We still get something very important out of this: the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ can be written as a combination of the basis vectors:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\vec{e}_1 + 1\vec{e}_2$$

When a linear transformation is applied, the way a vector is built from the basis vectors does not change, but *rather the direction of the basis vectors*.



(a) Identity vectors before transformation



(b) Identity vectors after transformation (no change)

Figure 1.1: An identity transformation acting on basis vectors.

A linear transformation acts on $\vec{x}$ by acting on each basis vector individually:

$$A\vec{x} = x_1 A\vec{e}_1 + x_2 A\vec{e}_2$$

such that any coefficient x_1 and x_2 are unchanged, while e_1 and e_2 are transformed. Thus, linear transformations preserve linear combinations while *altering the directions that define the coordinate system*. This will be better visualized in the next examples.

In the next example, we will apply a non-identity matrix and observe how transforming the basis vectors reshapes the entire space. Each figure will show a grid of points in $\mathbb{R}^2$ alongside the resulting transformed points after applying the matrix transformation. We encourage you to play with the python script `transforms.py` to see how different matrices affect the transformation. Also, we will limit our view to a square region from $(-2, -2)$ to $(2, 2)$ and only use 2×2 matrices for simplicity.

1.0.2 Scaling

Now, let's try scaling up the basis vectors. Consider the matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

This matrix will scale the e_1 vector by a factor of 1 (no change) and the e_2 vector by a factor of 2 (doubling its length).

Applying this transformation to the basis vectors produces this result:

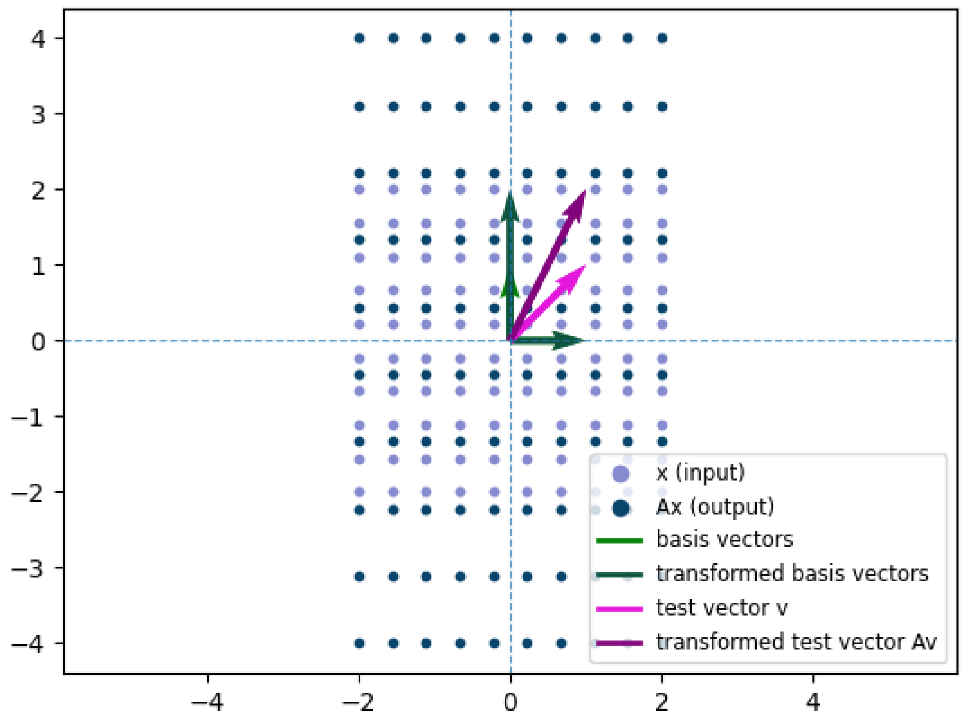


Figure 1.2: The effect of the scaling transformation $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ on the standard basis vectors.

1.1 Reflections

FIXME content not written just graphs across y (doesnt look different but x values change sign, e1 flips)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

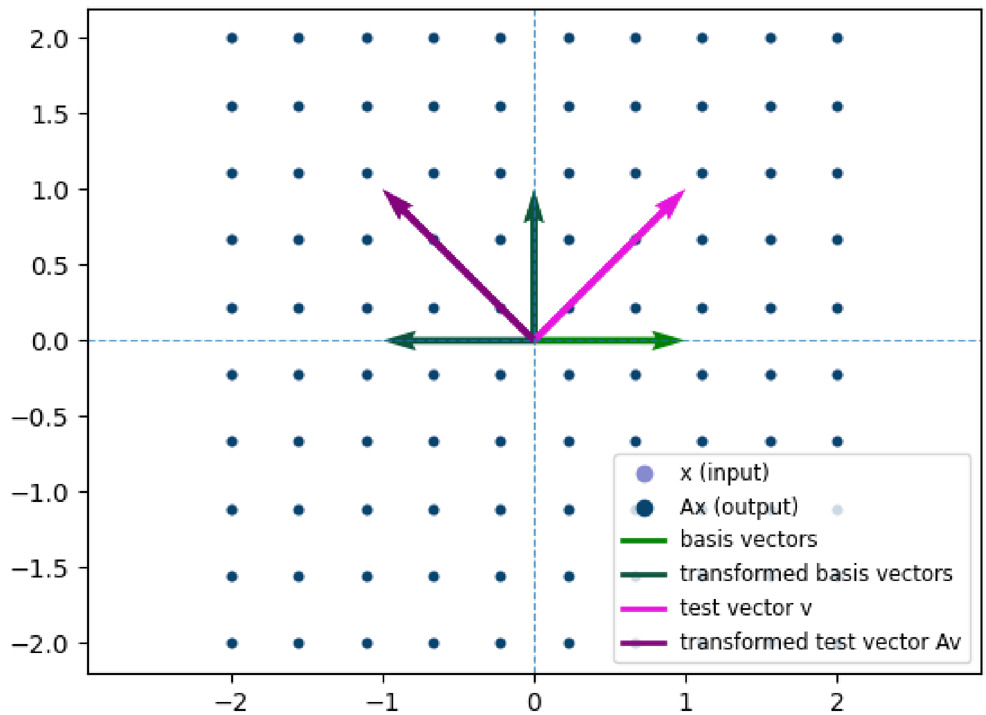


Figure 1.3: The effect of reflecting across the y-axis. Added is the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and its transformation in purple.

We can do the same with the x-axis. This gives us the matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

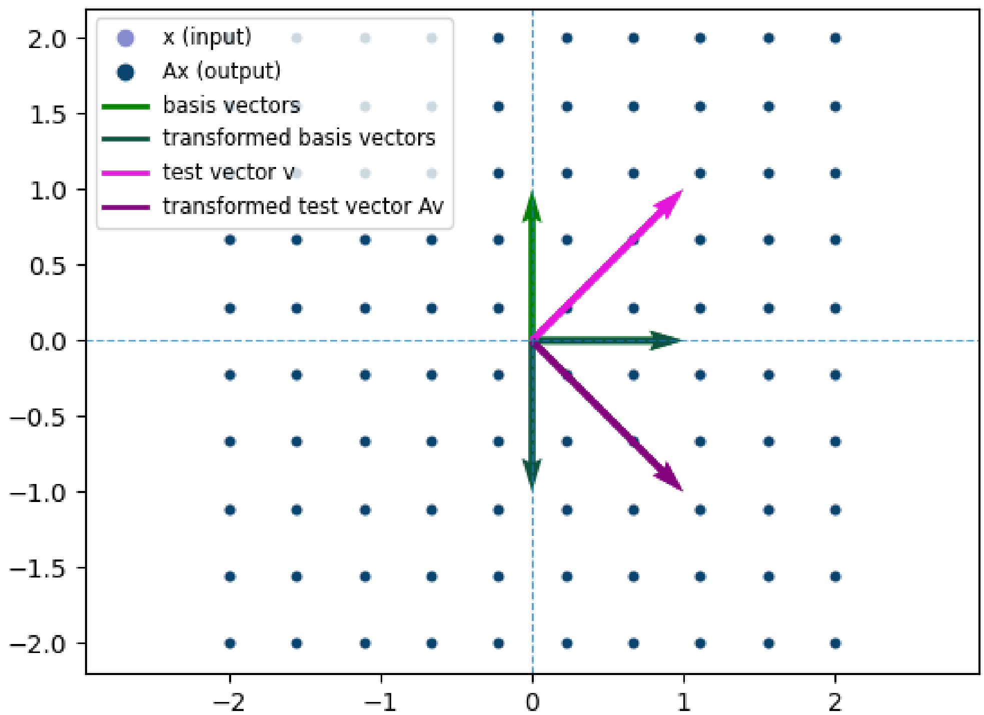


Figure 1.4: The effect of reflecting across the x-axis. Added is the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and its transformation in purple.

y = x and y = -x $(x, y) \mapsto (y, x)$ note that e1 and e2 swap places

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

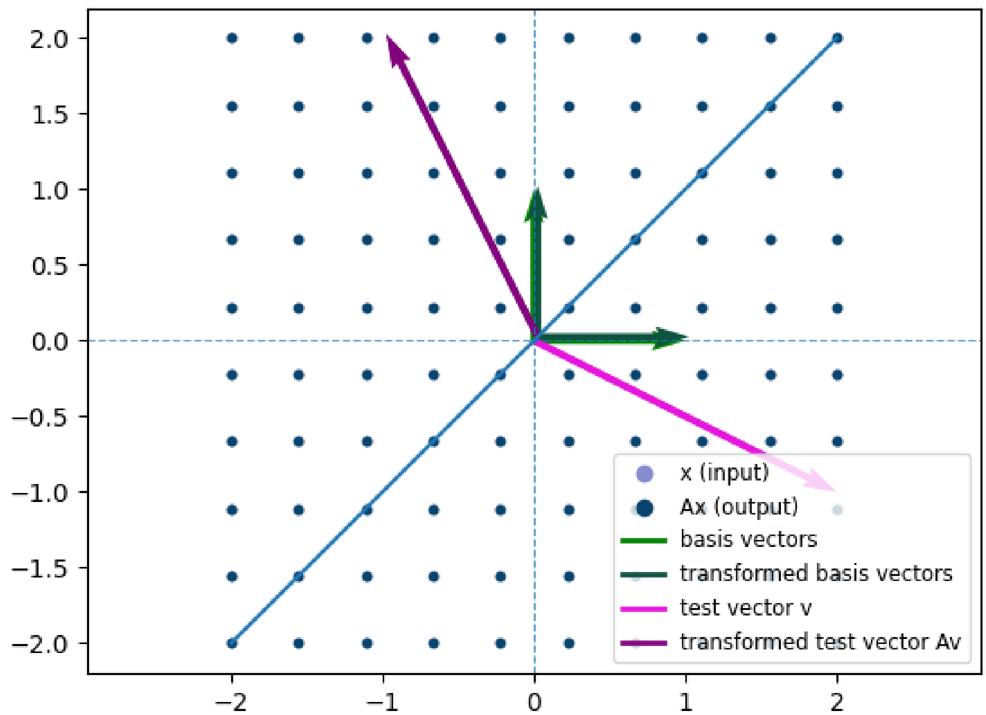


Figure 1.5: The effect of reflecting across the line $y = x$.

$(x, y) \mapsto (-y, -x)$ note that e1 and e2 swap places and flip signs

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

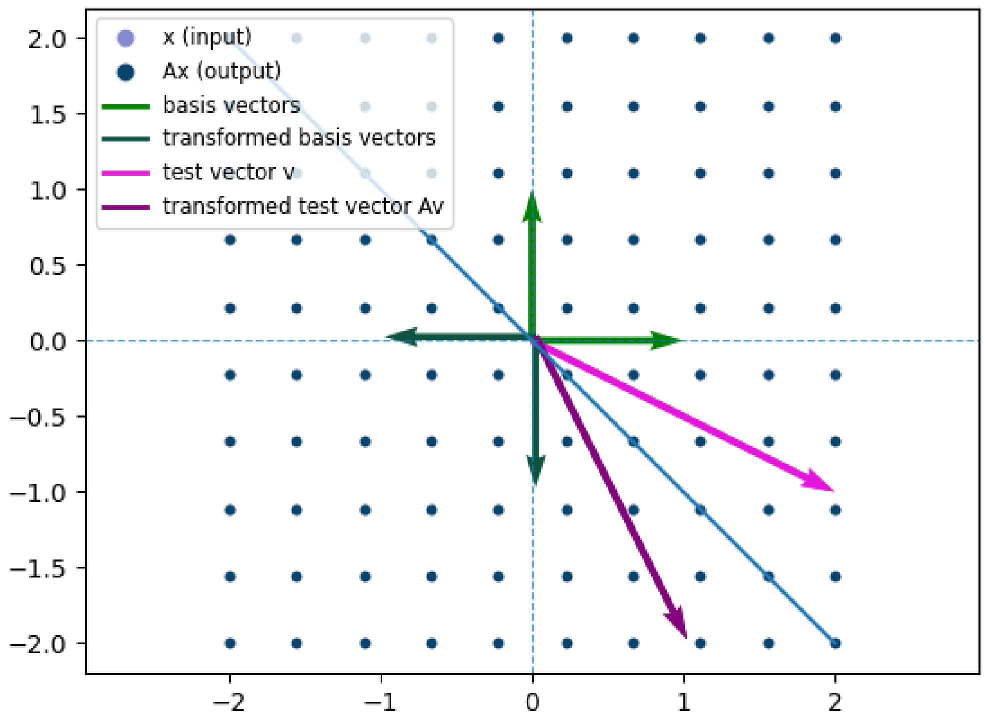


Figure 1.6: The effect of reflecting across the line $y = -x$.

1.2 Rotations

Rotation by 90 degrees ccw Rotation by 90 degrees cw

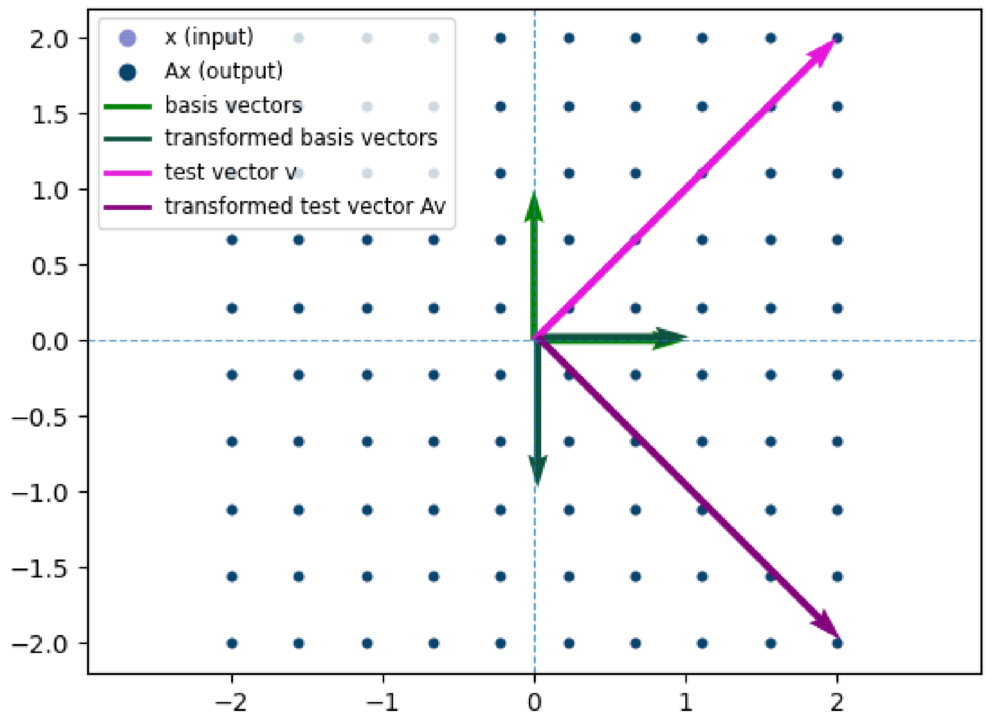


Figure 1.7: The effect of a 90 degree clockwise rotation.

Rotation ccw by theta rotation cw by theta

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`*) for more details.*

Appendix A

Answers to Exercises

Index