Permutations and Sorting

In the previous chapter, we talked about permutations. If you have a list of three letters, like [a, b, c, d], you can rearrange them in 4! ways:

a,b,c,d	a,b,d,c	a, d, b, c	a, d, c, b	a, c, b, d	a, c, d, b
b,a,c,d	b,a,d,c	b, d, a, c	b, d, c, a	b, c, a, d	b, c, d, a
c,b,a,d	c,b,d,a	c, d, b, a	c, d, a, b	c, a, b, d	c, a, d, b
d,b,c,a	d,b,a,c	d, a, b, c	d, a, c, b	d, c, b, a	d, c, a, b

You can make Python generate all the permutations for you:

```python
from itertools import permutations
all_permutations = permutations(('a', 'b', 'c', 'd'))
for p in all_permutations:
    print(p)
```

1.1 Notation

How do we define or write down a single permutation? You could say something like "Swap the first and second items and swap the third and fourth items." However, that gets pretty difficult to read. So we usually write a permutation as two lines: the first line is before the permutation and the second line is after. Like this:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

And we can assign permutations to variables. For example, if we wanted the variable A to represent "swapping the first and second item", we would write this:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

And if we wanted B to represent "swapping the third and fourth item", we would write:

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

Now, we can *compose* permutations together. For example, we might say:

$$B \circ A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

That is if we have the list $[a, b, c, d]$ and we apply permutation A and then permutation B, we get $[b, a, d, c]$.

Important: Note that permutations are applied from right to left. $B \circ A$ means "Applying A and then B." Why does this matter? Permutations are not necessarily commutative. That is, if you have two permutations S and T, $S \circ T$ is not always the same as $T \circ S$.

Also, note that "don't change anything" is a permutation. We call it *the identity permutation*. If you have four items, the identity permutation would be written:

$$I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

(We use a capital "I" for the identity.)

1.1.1   Challenge

Find an example of two permutations S and T such that $S \circ T$ does not equal $T \circ S$.

1.2   Sorting in Python

One of the common forms of permutation in software is sorting. Sorting is putting data in a particular order. For example, in Python, if you had a list of numbers, you can sort it in ascending order like this:

```
my_grades = [92, 87, 76, 99, 91, 93]
grades_worst_to_best = sorted(my_grades)
```

Do you want to sort backwards?

```
my_grades = [92, 87, 76, 99, 91, 93]
grades_best_to_worst = sorted(my_grades, reverse=True)
```

Note that sorted makes a new list with the correct order. If you want to sort the array in place, you can use the sort method:

```
my_grades = [92, 87, 76, 99, 91, 93]
my_grades.sort(reverse=True)
```

1.3 Inverses

Think for a second about this permutation:

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

You could say this permutation shuffles a list a bit. What is its inverse? That is, what is the permutation that unshuffles the items back to where they were originally?

$$S^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

That is, the original moved an item in the first spot to the third spot. The inverse must move whatever was in the third spot back to the first spot.

(Notation note: Because in multiplication, $b \times b^{-1} = 1$, we use "to the negative one" to indicate inverses in lots of places.)

Mechanically, how do you find the inverse? Flip the rows, and then sort the columns using the top number:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \text{ flip } \rightarrow \begin{pmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \text{ sort } \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

Let's say you have two permutations A and B. Permuting by B and then A would look like this:

$$C = A \circ B$$

If you know A^{-1} and B^{-1}, what is C^{-1}? You would undo-A and then undo-B, so

$$C^{-1} = B^{-1} \circ A^{-1}$$

1.4   Cycles

Here is a permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$$

When this is applied, whatever is at 1 gets moved to 2, 2 gets moved to 4, and 4 gets moved to 1. That is a *cycle*: $1 \rightarrow 2 \rightarrow 4$ and then it goes back to 1. It involves three locations, so we say it is a *3-cycle*.

There is another cycle in this permutation: $3 \rightarrow 5$ and then it goes back to 3.

Because these cycles share no members, we say the cycles are *disjoint*.

Every permutation can be broken down into a collection of disjoint cycles.

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} = (1 \rightarrow 2 \rightarrow 4)(3 \rightarrow 5)$$

The first handy thing about this notation is that it makes it easy for us to describe the inverse: we just run the cycles backward:

$$T^{-1} = (4 \rightarrow 2 \rightarrow 1)(5 \rightarrow 3)$$

Starting with the list $[a, b, c, d, e]$, lets repeatedly apply the permutation T

Initial	a, b, c, d, e
T applied	d, a, e, b, c
T ∘ T applied	b, d, c, a, e
T ∘ T ∘ T applied	a, b, e, d, c
T ∘ T ∘ T ∘ T applied	d, a, c, b, e
T ∘ T ∘ T ∘ T ∘ T applied	b, d, e, a, c
T ∘ T ∘ T ∘ T ∘ T ∘ T applied	a, b, c, d, e

This permutation, results in six combinations, and then it loops back on itself. The number of combinations is the least common multiple of all the cycles. In this case, there is a 3-cycle and a 2-cycle. The least common multiple of 2 and 3 is 6.

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Answers to Exercises

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