Chapter 1

Limits

The asymptotic behavior we see in rational functions suggests that we need to expand our vocabulary of function characteristics. We examined vertical asymptotes and end behavior through graphs and tables and discussed them in English. The language of limits enables us to discuss these attributes mathematically and with greater efficiency.

Let us revisit an example from the previous chapter. This function has a hole at $x = 1$, a vertical asymptote at $x = 3$, and a horizontal asymptote of $y = 1$.

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$$

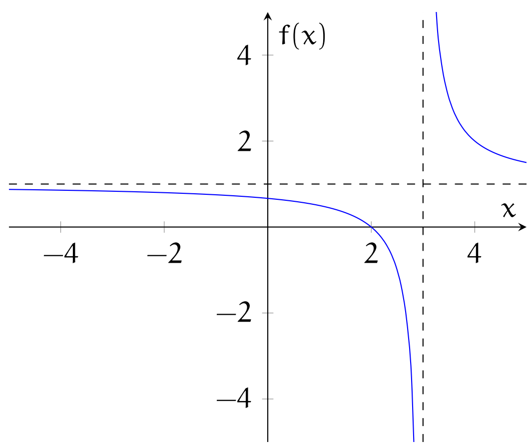


Figure 1.1: Graph of $f(x) = \frac{x^2-3x+2}{x^2-4x+3}$ with asymptotes

First, consider the vertical asymptote. We see that the graph goes down as it hugs the left side of the vertical asymptote, and goes up as it hugs the right side. We can describe these behaviors as the left- and right-hand limits, respectively. We say that the left-hand limit of f at $x = 3$ is negative infinity. Another way of communicating this is to say that as x approaches 3 from the left, the function approaches negative infinity. Symbolically, we summarize this as

$$\lim_{x \to 3^-} f(x) = -\infty$$

The little negative sign in $x \to 3^-$ indicates we are approaching $x = 3$ from the left (the negative side of the axis).

Similarly, the right-hand limit of f at $x = 3$ is positive infinity. In other words, as x

approaches 3 from the right, the function approaches positive infinity. Symbolically, we write

$$\lim_{x \to 3^+} f(x) = \infty$$

This time, the little $^+$ indicates we are approaching the x-value from the right (positive) side of the axis.

The limit of a function at a particular x-value is the y-value that the function approaches as it approaches the given x-value. In the previous example, we could only specify the left- and right-hand limits, because they were different. In cases where the left- and right-hand limits are equal, we can say that the function has a limit there. The hole in our function f is one such value. We see that as we approach the hole from both the left and right, the function takes on values near $\frac{1}{2}$. This is more apparent numerically:

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	0.5238	0.5025	0.5003	undefined	0.4998	0.4975	0.4737

We can also see this by zooming in on the graph (see figure ??):

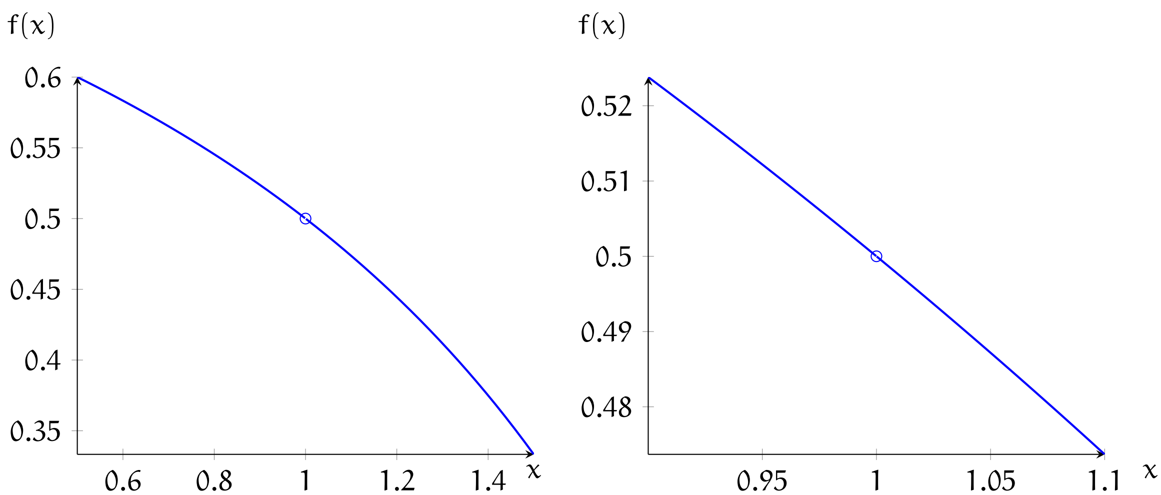


Figure 1.2: Two graphs of $f(x) = \frac{x^2-3x+2}{x^2-4x+3}$ zoomed in about $x = 1$

The left-hand and right-hand limits of f at 1 are both $\frac{1}{2}$. Since they are equal, we can also say that the limit of f at 1 is $\frac{1}{2}$. This allows us to efficiently discuss the behavior of f at 1, even though the function is not defined there since substituting 1 into the function gives division by zero.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = \frac{1}{2}$$

We can also talk about limits at x-values where nothing weird is happening, that is, no hole or vertical asymptote. For example, as x approaches 4 from the left and right, y approaches 2.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	2.1111	2.0101	2.0010	2	1.9990	1.9901	1.9091

In this case, since nothing weird is happening, the limit is equal to the function value. This is an example of continuity, which we will discuss in more detail in the next chapter. By contrast, at the vertical asymptote $x = 1$, since the left- and right-hand limits are not equal, we say the function does not have a limit, or the limit does not exist.

Finally, let us consider the horizontal asymptote of f. The graph hugs the line $y = 1$ as x goes far to the left and far to the right. We say that as x approaches negative infinity, f approaches 1, and likewise, that as x approaches positive infinity, f approaches 1. We write these symbolically as $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 1$.

Exercise 1 Limits Practice 1

Determine the left- and right-hand limits of the function as x approaches the given values. At x-values where the limit exists, determine it.

1. $p(x) = \frac{x+3}{x^2+9x+18}, x = -6, -5, -3, \infty$

Working Space

Answer on Page 25

We have seen two weird behaviors of rational functions at certain x-values: holes and vertical asymptotes. Now we will examine another type of weird behavior: jumps. This is a characteristic of some piecewise defined functions. In piecewise defined functions, the domain is divided into two or more pieces, and a different expression is used to give the y-value depending on which piece contains the x- value. One common piecewise defined function is the floor function (shown in figure 1.3), sometimes denoted $\lfloor x \rfloor$. The standard floor function rounds any real number down to the nearest integer. So, for a price quoted in dollars and cents, the floor would be just the number of dollars.

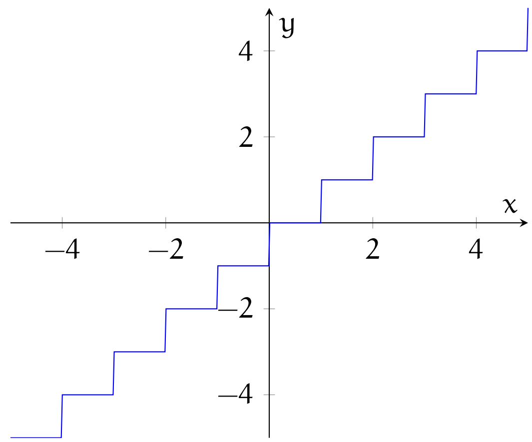


Figure 1.3: Graph of $y = \lfloor x \rfloor$

When x is exactly 1, the function value is 1: the number of dollars in a price of \$1.00. When x is any number greater than 1 but less than 2, the function value is still 1. Also, $\lfloor 1.01 \rfloor, \lfloor 1.5 \rfloor, \text{and} \lfloor 1.99999 \rfloor$ are all 1. As we continue to look to the right, once x equals exactly 2, h jumps up to the value 2. So, $\lim_{x \to 2^-} \lfloor x \rfloor = 1$, while $\lim_{x \to 2^+} \lfloor x \rfloor = 2$.

Besides rational and piecewise defined functions, there are other functions with interesting limits. Consider the standard exponential function, $y = e^x$ (shown in figure 1.4).

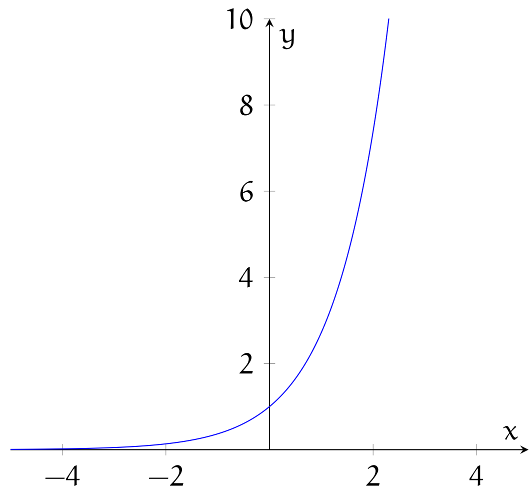


Figure 1.4: Graph of $y = e^x$

As x increases, y increases without bound; that is, $\lim_{x \to \infty} e^x = \infty$. However, looking far to the left, we see that y hugs the x-axis. This is because raising e to a large negative exponent is the same as 1 divided by e raised to a large positive exponent; that is, 1 divided by a very large number, which yields a very small positive number. In limit notation, $\lim_{x\to-\infty} e^x = 0$. This example illustrates that horizontal asymptotes need not model end behavior in both directions. Note that this reasoning holds for $y = b^x$ for any $b > 1$, so all such functions have the same horizontal asymptote, $y = 0$.

We know that the natural logarithm function, $y = \ln x$, is the inverse of $y = e^x$. Since inverse functions swap the role of x and y, it stands to reason that a horizontal asymptote in one function corresponds with a vertical asymptote in the other function, and that is indeed the case (see figure 1.5).

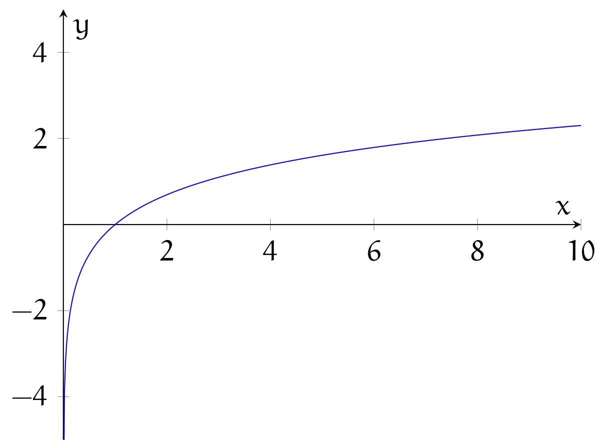


Figure 1.5: Graph of $y = \ln x$

An untransformed logarithm function is defined only for positive inputs. That is because it is not possible to find an exponent of a positive number which will yield a negative or zero result. What type of exponent on a positive number yields a number near zero? That would be a large-magnitude negative number. So, on the logarithm graph, large negative y-values correspond with x-values only slightly greater than zero. So, $\ln x$ (and $\log_2 x$, and indeed $\log_b x$ for any $b > 1$) approaches negative infinity as x approaches 0 from the right. There is no left-hand limit at 0, however. In limit notation, $\lim_{x\to 0^+} \ln x = -\infty$.

Exercise 2 Limits Practice 2

State the asymptotes of the following transformed exponential and logarithmic functions. Give the limit statement which describes the behavior of the function along the asymptote.

1. $y = 3^x + 1$
2. $y = \log_2(x - 4)$
3. $y = 2^{1-x}$
4. $y = \log_{10}(-2x)$

Working Space

Answer on Page 25

We next consider two functions which each have two horizontal asymptotes. These two seemingly obscure functions are quite important in data science.

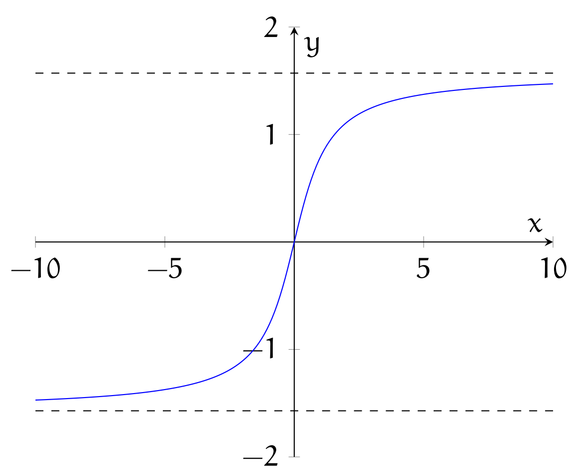


Figure 1.6: Graph of $y = \arctan x$

We know that the arctangent, or inverse tangent, function is the inverse of the piece of the tangent function which passes through the origin. The vertical asymptotes bounding this piece become horizontal asymptotes when the function is inverted.

Here are the equation and graph of the logistic function:

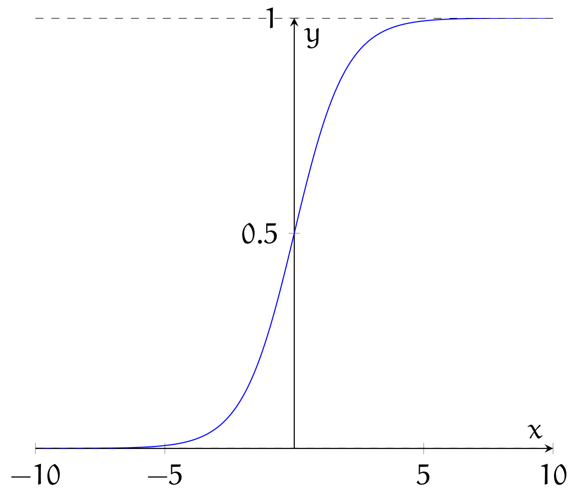


Figure 1.7: Graph of the logistic function, $y = \frac{1}{1+e^{-x}}$

For large magnitude negative values of x, the exponential term in the denominator becomes a very large positive value. The fraction thus becomes a positive number very close to zero. For large magnitude positive values of x, that exponential term becomes a very small positive number. Adding it to 1 yields a denominator just barely greater than 1. Dividing 1 by this number thus yields a function value just barely less than 1. So, the logistic function yields values between 0 and 1, though never equaling either of these values exactly. It is precisely this characteristic which makes the logistic function so useful.

Exercise 3 Limits Practice 3

Using limit notation, state the limits as x approaches negative and positive infinity for the inverse tangent and logistic functions given above.

Working Space

Answer on Page 25

As seen above, the limit of a function from the left may be different from the limit of the function from the right. Additionally, the actual *value* of the function may be different from the limit. Consider the piecewise function $h(x)$:

$$h(x) = \begin{cases} -x^2+3, & \text{if } x<0 \\ 2, x=0 \\ -x+3, & \text{if } x>0 \end{cases}$$

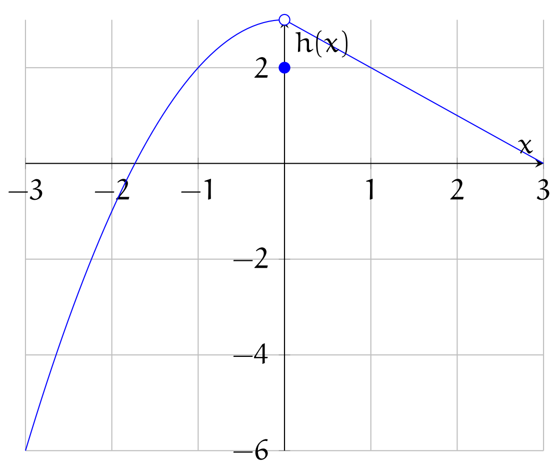


Figure 1.8: Graph of the piecewise function, $h(x)$

From examining the graph, we see that

$$\lim_{x \to 0_-} h(x) = \lim_{x \to 0_+} h(x) = 3$$

However, $h(0) = 2 \neq 3$. So, does this limit exist? It does! The limit of a function describes the *behavior* of the function around a particular value, not the value of the function itself. In order for a limit to exist, the limits from the left and right must be equal to each other, but not necessarily the actual value of the function.

Use the graph of $h(x)$ above and the graphs of $f(x)$ and $g(x)$ below to complete the following exercise.

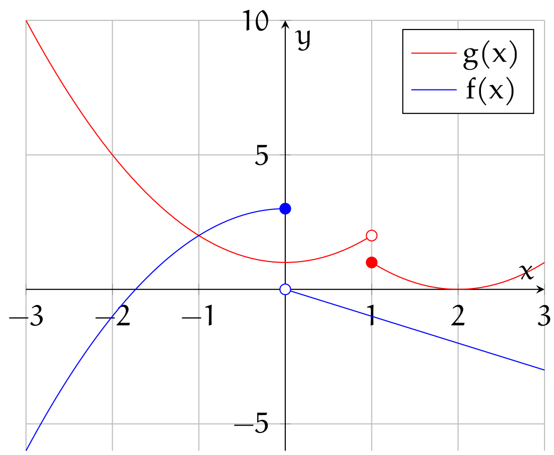


Figure 1.9: Piecewise functions $f(x)$ and $g(x)$

Exercise 4 Limits Practice 4

Determine the limit from the left and the right for each function at the given value(s). State the limit at that value, if it exists.

1. $h(x), x = -1, 0, 1$
2. $f(x), x = -1, 0, 2$
3. $g(x), x = -2, 0, 1, 2$

Working Space

Answer on Page 25

1.1 Continuity

A note about continuity:

In order to be able to talk more about limits and know when we can apply certain rules and theorems, we first must discuss continuity. A function is continuous if there are no "jumps" or "gaps" in the graph of the function. For example, the function $f(x) = x^2$ is continuous for all real values of x. On the other hand, the function $g(x) = \tan(x)$ has many discontinuities, including at $x = \frac{\pi}{2}$. Let's examine the graph of each of these functions:

If you wanted, you could trace your finger along the graph of $f(x)$ from $x = -3$ to $x = 3$ without ever picking up your finger. This means the function is continuous in the domain from $-3 \leq x \leq 3$. In this case, the domain of continuity *includes* the end points ($x = 3$ and

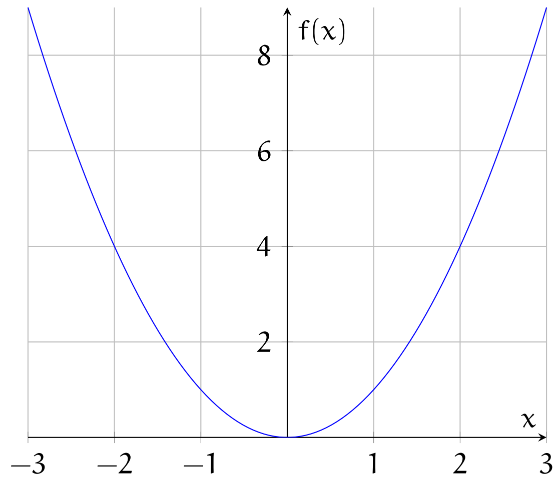


Figure 1.10: Graph of $f(x) = x^2$

$x = -3$). This is called a closed interval. In other cases, the function will be continuous right up to, but not including, the endpoints, as with the domains of continuity for our other example, $g(x) = \tan x$. This is called an open interval. Let's learn more about intervals of continuity by examining $g(x) = \tan x$.

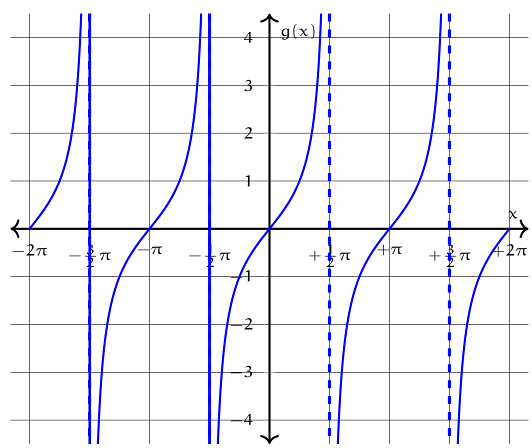


Figure 1.11: Graph of $g(x) = \tan x$

As you can see, if you trace your finger along the graph of the function starting at $x = 0$, you can continue without lifting your finger to $x = \frac{\pi}{2}$. As you approach $x = \frac{\pi}{2}$ from the left, the value of $g(x)$ approaches ∞. In order to continue tracing the function PAST $x = \frac{\pi}{2}$, you have to lift your finger and bring it down to $-\infty$. The function then continues continuously again until $x = \frac{3\pi}{2}$.

In the case of $g(x) = \tan x$, the function is continuous on *open intervals*, including the open interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

There is a shorter way to represent open and closed domain intervals. We can represent that $f(x) = x^2$ is continuous on the closed interval $-3 \leq x \leq 3$ in the following way:

$$x \in [3, -3]$$

Which reads as "x contained in the domain -3 to 3, inclusive". That is, all the values from -3 to 3, including the endpoints. The inclusion of the endpoints is implied by the use of *brackets*. For open intervals, we use parentheses to communicate that the interval goes up to, but does not include, the endpoints. For $g(x) = \tan x$, we can use parentheses:

$$x \in \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$$

because the $g(x) = \tan x$ is not continuous at $x = \frac{-3\pi}{2}$ or at $x = \frac{-\pi}{2}$.

Formally, a function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists *and* $\lim_{x \to a} f(x) = f(a)$. That is, the limit is equal to the actual value of the function. Re-examine the graph of $h(x)$. We have already seen that $\lim_{x \to 0} h(x)$ exists and is equal to 3. However, $h(0) = 2 \neq \lim_{x \to 0} h(x)$. So $h(x)$ is not continuous at $x = 0$. Because $-x^2 + 3$ is evaluable all the way to $-\infty$ and $-x + 3$ is evaluable all the way to ∞, the function $h(x)$ is continuous everywhere *except* $x = 0$. We can represent this mathematically by saying $h(x)$ is continuous on the domain $x \in (-\infty, 0) \cup (0, \infty)$. We use parentheses for $\pm\infty$ because we can never actually reach ∞. Additionally, the function is continuous up to, but not including 0, and the use of parentheses excludes $x = 0$ from the domain of continuity.

1.1.1 Continuity Practice

Exercise 5

Working Space

[This problem was originally presented as a calculator-allowed, multiple- choice question on the 2012 AP Calculus BC exam.] Suppose a function f is continuous at $x = 3$. Classify the following statements at always true, sometimes true, or never true. Explain your answers.

1. $f(3) < \lim_{x \to 3} f(x)$
2. $\lim x \to 3^{+} f(x) \neq \lim_{x \to 3^{-}} f(x)$
3. $f(3) = \lim_{x \to 3^{+}} f(x) = \lim_{x \to 3^{-}} f(x)$
4. The derivative of f at $x = 3$ exists.
5. The derivative of f is positive for $x < 3$ and negative for $x > 3$.

Answer on Page 26

Exercise 6 Limits Practice 5

Working Space

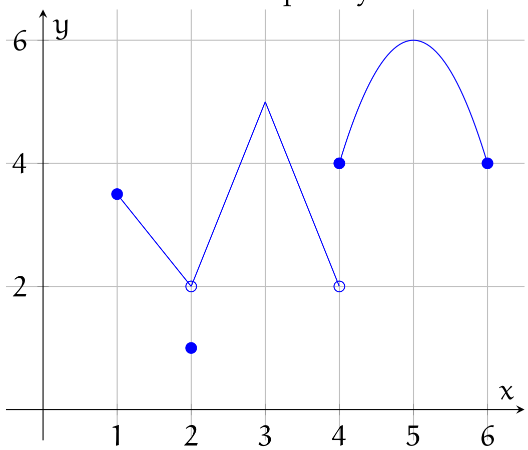
State the location of discontinuities (if any) and explain why the function is discontinuous at that location:

1. $f(x) = \frac{3x^2 - 8x - 3}{x - 3}$
2. $f(x) = \begin{cases} \frac{2}{x^4}, & \text{if } x <\neq 0 \\ 2, & \text{if } x = 0 \end{cases}$
3. $f(x) = \begin{cases} \frac{3x^2 - 8x - 3}{x - 3}, & \text{if } x \neq 3 \\ 1, & \text{if } , x = 3 \end{cases}$

Answer on Page 26

Exercise 7

The graph of a function, $h(x)$, is shown. Classify each of the following statements as true or false and explain your answer.



1. $\lim_{x\to 2} h(x)$ exists
2. $\lim_{x\to 3} h(x)$ does not exist
3. $\lim_{x\to 4} h(x)$ exists
4. $h(x)$ is continuous at $x = 5$
5. $h(x)$ is not continuous at $x = 4$
6. $\lim_{x\to 2} h(x) = h(2)$

Working Space

Answer on Page 27

1.2 Limits Rules

There are some mathematical properties of limits which allow us to determine the limit of complex functions without seeing a graph or using a calculator to generate a table.

The following laws are true given that c is a constant, $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} g(x)$ exists.

1. Sum Law $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. Difference Law $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. Constant Multiple Law $\lim_{x\to a}[cf(x)] = c \cdot \lim_{x\to a} f(x)$
4. Product Law $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. Quotient Law $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ given that $\lim_{x\to a} g(x) \neq 0$

These laws are fairly obvious - the limit of the sum of two functions is equal to the sum of the limits of each function individually. The only tricky on is the last: the limit of the quotient of two functions is equal to the quotient of the limits if and only if the limit of the function in the denominator does not equal zero. This makes sense, since we know dividing by zero yields an undefined result.

Let's practice applying these laws to evaluate the limits of the functions $f(x)$, shown in blue below, and $g(x)$, shown in red below:

$$f(x) = \begin{cases} -x^2+3, & \text{if } x \leq 0 \\ -x, & \text{if } x > 0 \end{cases}$$

$$g(x) = \begin{cases} x^2+1, & \text{if } x < 1 \\ (x-2)^2, & \text{if } x \geq 1 \end{cases}$$

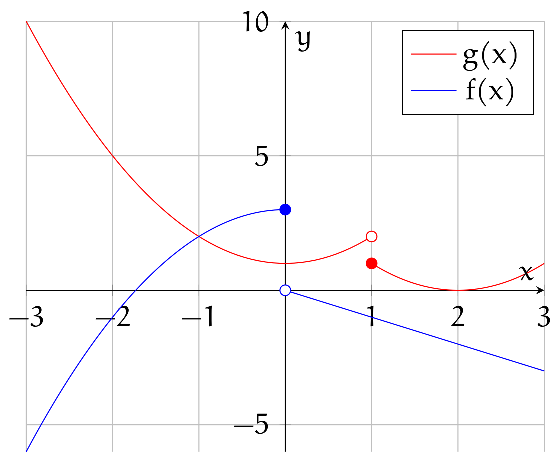


Figure 1.12: Graphs of the piecewise functions $f(x)$ and $g(x)$

We can use these laws to evaluate limits involving $f(x)$ and $g(x)$ (shown on the graph above). Here are some examples: Use the graphs of $f(x)$ and $g(x)$ given above to evaluate each limit, if it exists. If the limit does not exist, explain why. Two examples are given first:

Example 1: Evaluate $\lim_{x\to 0} f(x) \cdot g(x)$

Solution 1: From the Product Law, we know that:

$$\lim_{x\to 0} f(x)\cdot g(x) = \lim_{x\to 0} f(x)\cdot \lim_{x\to 0} g(x)$$

Looking at the graph, we can see that

$$\lim_{x\to 0} g(x) = 1$$

and there is a discontinuity in $f(x)$ at $x = 0$. Therefore,

$$\lim_{x\to 0} f(x) = \text{undef}$$

Substituting this, we get:

$$\lim_{x\to 0} f(x)\cdot \lim_{x\to 0} g(x) = \text{undef}\cdot 1 = \text{undef}$$

Therefore, *the limit does not exist*.

Example 2: Evaluate $\lim_{x\to 2} f(x) - g(x)$

Solution 2: Applying the Difference Law, we see that:

$$\lim_{x\to 2}[f(x) - g(x)] = \lim_{x\to 2} f(x) - \lim_{x\to 2} g(x)$$

Examining the graph, we see that

$$\lim_{x\to 2} f(x) = -2$$

and

$$\lim_{x\to 2} g(x) = 0$$

Substituting these values, we get:

$$\lim_{x\to 2} f(x) - g(x) = -2 - 0 = -2$$

Exercise 8 Limits Practice 6

Working Space

1. $\lim_{x\to-3} \frac{f(x)}{g(x)}$
2. $\lim_{x\to 2} [f(x) + 5g(x)]$
3. $\lim_{x\to-1} \frac{3g(x)}{f(x)}$
4. $\lim_{x\to 0} f(x) \cdot 5g(x)$
5. $\lim_{x\to-1} f(x) - 3g(x)$

Answer on Page 27

Recall that exponents represent repeated multiplication. Therefore, if we apply the Product Law multiple times, we obtain the Power Law for limits:

6. Power Law $\lim_{x\to\infty} [f(x)]^n = [\lim_{x\to\infty} f(x)]^n$ where n is a positive integer

There are two special limits that will be useful to us which are intuitively obvious, but we won't formally prove here.

7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$

Combining Law 8 with the Power Law, we find that:

9. $\lim_{x\to a} x^n = a^n$

And similarly, for square roots:

10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (if n is even, we assume $a > 0$)

Direct substitution property: If f is a polynomial or rational function and a is in the domain for f, then

$$\lim_{x\to a} f(x) = f(a)$$

Often, rational functions can be simplified. In an above example, we computed the limit by simplifying $f(x) = \frac{3x^2-8x-3}{x-3}$ to the simpler $g(x) = 3x+1$. This is a valid strategy because $\frac{3x^2-8x-3}{x-3} = 3x+1$ when $x \neq 3$. Remember: a limit describes how a function behaves *as it approaches* a, not its value/behavior when x *actually equals* a. This reveals the following useful rule:

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limit exists.

1.3 Squeeze Theorem

The Squeeze Theorem states that if $f(x) \leq g(x) \leq h(x)$ when x is near a (except at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

In words, if $g(x)$ is between $f(x)$ and $h(x)$ near a and f and h have the same limit, L, then the limit of g must also be L.

Example: Let's examine the graph of $g(x) = x^2 \sin\frac{1}{x}$ and determine $\lim_{x\to 0} g(x)$:

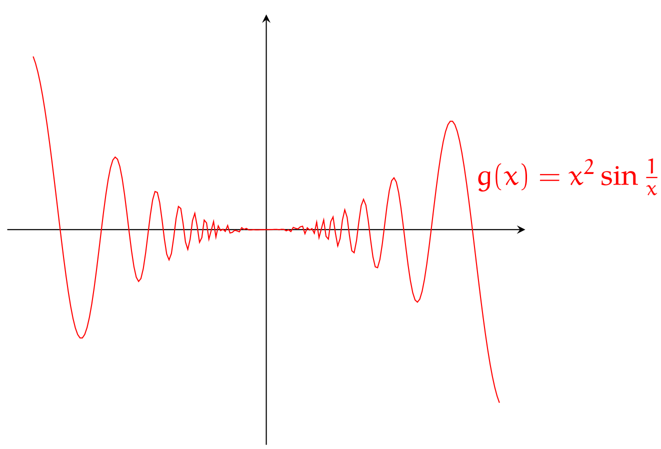


Figure 1.13: Graph of $g(x) = x^2 \sin\frac{1}{x}$

Solution: Because $\sin\frac{1}{x}$ is undefined at $x = 0$, we cannot compute the limit directly. However, from examining the graph, we can guess that $\lim_{x\to 0} g(x) = 0$. Feel free to confirm this with your calculator. We need to choose two functions: one that is larger than $g(x)$ near $x = 0$ and one that is smaller. Since $|\sin\frac{1}{x}| \leq 1$ (when $x \neq 0$), then

$$|x^2 \sin\frac{1}{x}| \leq x^2$$

and

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

Let's confirm this by plotting $f(x) = -x^2$, $g(x)$, and $h(x) = x^2$ on the same graph:

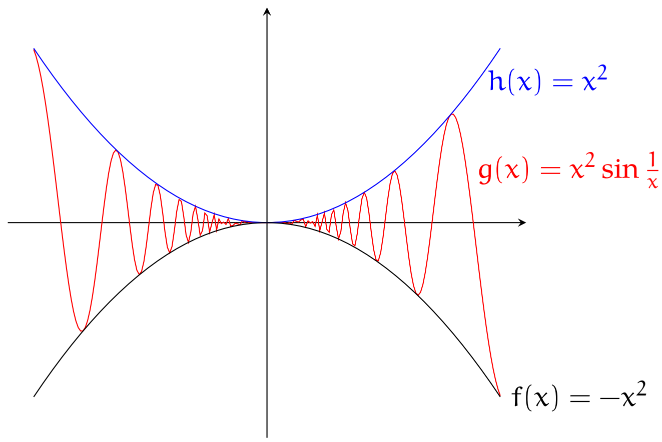


Figure 1.14: Squeeze Theorem example

As you can see, when x is near 0, $f(x) \leq g(x) \leq h(x)$. Because $f(x)$ and $h(x)$ are both polynomials, their limits are straightforward:

$$\lim_{x\to 0} -x^2 = 0 \text{ and } \lim_{x\to 0} x^2 = 0$$

Then, by the Squeeze Theorem, we can say that:

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

1.3.1 Squeeze Theorem Practice

Exercise 9 Squeeze Theorem 1

Use the Squeeze Theorem to show that $\lim_{x\to 0} \sqrt{x^3 + x^2} \cos \frac{1}{x} = 0$. Illustrate by graphing the functions you define as f, g, and h on the same plot.

Working Space

Answer on Page 29

Exercise 10 **Squeeze Theorem 2**

If $2x + 3 \leq f(x) \leq x^2 - 2x + 7$ for $x \geq 0$, find $\lim_{x \to 2} f(x)$

Working Space

Answer on Page 29

Exercise 11 **Squeeze Theorem 3**

Prove that $\lim_{x \to 0^+} \sqrt{x} e^{\sin \frac{\pi}{x}} = 0$

Working Space

Answer on Page 29

1.4 Intermediate Value Theorem

When considering functions that are continuous on a closed interval, the Intermediate Value Theorem can help us: Given a function, $f(x)$, that is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$, there is at least one number c such that $f(c) = N$, where N is any number between $f(a)$ and $f(b)$. The theorem is illustrated in figures 1.15 and 1.16:

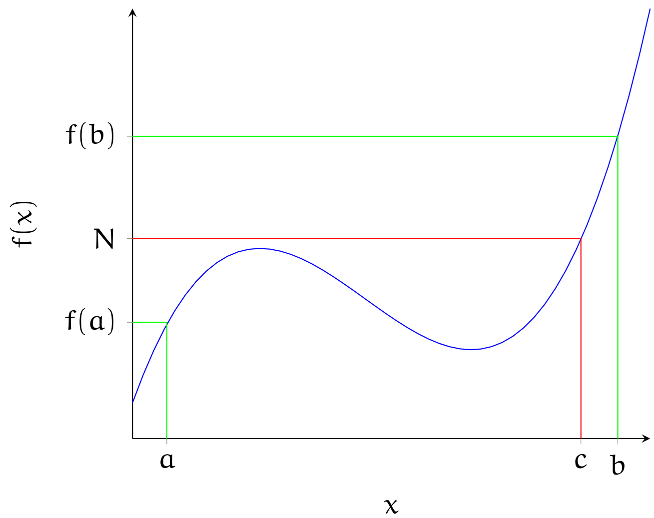


Figure 1.15: An example where one solution satisfies the Intermediate Value Theorem

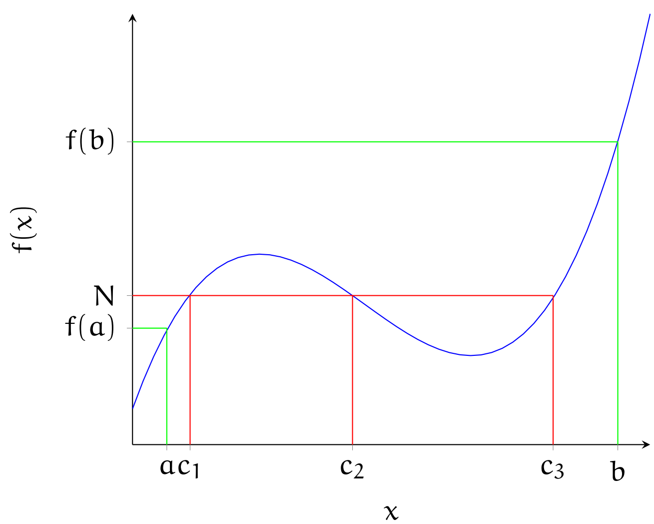


Figure 1.16: An example where more than one solution satisfies the Intermediate Value Theorem

Logically, we can think of the IVT this way: if a function is continuous on a closed interval, there are no gaps or breaks. If there are no gaps or breaks, then the function must pass through the line $y = N$, since it cannot jump over the line. Your graphing calculator uses IVT to find roots of functions:

Example: show that there is at least one root of the equation $2x^3 - 6x^2 + 3x - 1 = 0$ between $x = 2$ and $x = 3$.

Solution: For IVT to apply, we must first check that the function is continuous on the closed interval $x \in [2, 3]$. We define $f(x) = 2x^2 - 6x^2 + 3x - 1$, which is continuous everywhere, because it is a polynomial function. *For more complex functions, always be sure to check the endpoints of an interval, since IVT only applies on closed intervals of continuity*. We will take $a = 2$, $b = 3$, and $N = 0$. We find the values of $f(x)$ at the endpoints:

$$f(2) = 2(2)^3 - 6(2)^2 + 3(2) - 1 = 16 - 24 + 6 - 1 = -3$$
$$f(3) = 2(3)^3 - 6(3)^2 + 3(3) - 1 = 54 - 54 + 9 - 1 = 8$$

Therefore, $f(2) < 0 < f(3)$ and according to IVT, there must exist some c such that $f(c) = 0$ and there is a root to the equation in the interval $x \in (2, 3)$.

Exercise 12 Intermediate Value Theorem Practice

Use the IVT to show there is a solution the given equation on the stated interval:

1. $2x^4 + x - 12 = 0$, $(1, 2)$
2. $\ln(x) = 3x - 4\sqrt{x}$, $(2, 3)$
3. $2\sin x = 3x^2 - 2x$, $(1, 2)$

Working Space

Answer on Page 30

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 3)

$$\lim_{x\to-6^-} p(x) = -\infty, \lim_{x\to-6^+} p(x) = \infty$$

$$\lim_{x\to-5^-} p(x) = \lim_{x\to-5^+} p(x) = \lim_{x\to-5} p(x) = 1$$

$$\lim_{x\to-3^-} p(x) = \lim_{x\to-3^+} p(x) = \lim_{x\to-3} p(x) = \frac{1}{3}$$

$$\lim_{x\to\infty} p(x) = 0 \text{ called simply a limit, although it is a left-hand limit}$$

Answer to Exercise 2 (on page 6)

$$\lim_{x\to-\infty} 3^x + 1 = 1; \lim_{x\to4^+} \log_2(x-4) = -\infty; \lim_{x\to\infty} 2^{1-x} = 0; \lim_{x\to0^-} \log_{10}(-2x) = -\infty$$

Answer to Exercise 3 (on page 8)

$\lim_{x\to-\infty} \tan^{-1}x = -\frac{\pi}{2}, \lim_{x\to\infty} \tan^{-1}x = \frac{\pi}{2}; \lim_{x\to-\infty} \frac{1}{1+e^{-x}} = 0, \lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1$

Answer to Exercise 4 (on page 10)

1. $\lim_{x\to-1_-} h(x) = 2$ and $\lim_{x\to-1_+} h(x) = 2$, therefore the limit exists and $\lim_{x\to-1} h(x) = 2$

   $\lim_{x\to0_-} h(x) = 3$ and $\lim_{x\to0_+} h(x) = 3$, therefore the limit exists and $\lim_{x\to0} h(x) = 3$

   $\lim_{x\to1_-} h(x) = 2$ and $\lim_{x\to1_+} h(x) = 2$, therefore the limit exists and $\lim_{x\to1} h(x) = 2$

2. $\lim_{x\to-1_-} f(x) = 2$ and $\lim_{x\to-1_+} f(x) = 2$, therefore the limit exists and $\lim_{x\to-1} f(x) = 2$.

$\lim_{x\to 0_-} f(x) = 3$ and $\lim_{x\to 0_+} f(x) = 0$, and because $\lim_{x\to 0_-} f(x) \neq \lim_{x\to 0_+} f(x)$, the limit does not exist.

$\lim_{x\to 2_-} f(x) = -2$ and $\lim_{x\to 2_+} f(x) = -2$, therefore the limit exists and $\lim_{x\to 2} f(x) = -2$.

3. $\lim_{x\to -2_-} g(x) = -1$ and $\lim_{x\to -2_+} g(x) = -1$, therefore the limit exists and $\lim_{x\to -2} g(x) = -1$.

   $\lim_{x\to 0_-} g(x) = 1$ and $\lim_{x\to 0_+} g(x) = 1$, therefore the limit exists and $\lim_{x\to 0} g(x) = 1$

   $\lim_{x\to 1_-} g(x) = 2$ and $\lim_{x\to 0_+} g(x) = 1$, and because $\lim_{x\to 1_-} g(x) = 2 \neq \lim_{x\to 0_+} g(x)$, the limit does not exist.

   $\lim_{x\to 2_-} g(x) = 0$ and $\lim_{x\to 2_+} g(x) = 0$, therefore the limit exists and $\lim_{x\to 2} g(x) = 0$

Answer to Exercise 5 (on page 13)

1. Never true. If a function is continuous at a, then $f(a) = \lim_{x\to a} f(x)$.
2. Never true. If a function is continuous at a, then $\lim x \to a^+ f(x) = \lim x \to a^- f(x)$
3. Always true. This is the definition of continuity.
4. Sometimes true. The derivative of f at $x = 3$ exists for $f(x) = x^2$ but not for $f(x) = |x-3|$.
5. Sometimes true. This statement is true for $f(x) = -(x-3)^2$ but not for $f(x) = 4x$.

Answer to Exercise 6 (on page 13)

1. $f(x)$ is not defined at $x = 3$, therefore it is also discontinuous at $x = 3$. As we learn about the continuity of polynomials, we'll see why $f(x)$ is continuous everywhere else.
2. Here, $f(0)$ is defined, so we need to check if $\lim_{x\to 0} f(x) = f(0)$. The left and right limits as x approaches 0 are the same (∞), so the limit exists. However, $f(0) = 1 \neq \lim_{x\to 0} f(x)$. Therefore, the function is discontinuous at $x = 0$.
3. In this function, $f(3)$ is defined, so we need to check if the limit equals the function value. The limit of $f(x)$ as x approaches 3 is:

$$\lim_{x\to 3} \frac{3x^2 - 8x - 3}{x-3} = \lim_{x\to 3} \frac{(3x+1)(x-3)}{x-3} = \lim_{x\to 3} 3x + 1 = 10$$

   So the limit exists, but $\lim x \to 3 f(x) \neq f(3)$ and we see that the function is discontinuous at $x = 3$.

Answer to Exercise 8 (on page 17)

1. True, $h(x)$ approaches 2 from the left and right, therefore the limit exists
2. False, $h(x)$ approaches 5 from the left and right, therefore the limit exists
3. False, $h(x)$ approaches 2 from the left and 4 from the right, therefore the limit does not exist
4. True, $\lim_{x \to 5} h(x) = h(5)$, therefore $h(x)$ is continuous at $x = 5$
5. True, $\lim_{x \to 4} h(x)$ does not exist, therefore $h(x)$ is discontinuous at $x = 4$
6. False, $h(2) = 1 \neq 2 = \lim_{x \to 2} h(x)$

Answer to Exercise 8 (on page 17)

1. From the quotient law, we know that:

   $$\lim_{x \to 3} \frac{f(x)}{g(x)} = \frac{\lim_{x \to 3} f(x)}{\lim_{x \to 3} g(x)}$$

   From the graph, we see that:

   $$\lim_{x \to 3} f(x) = -3$$

   and that:

   $$\lim_{x \to 3} g(x) = 1$$

   Substituting these values, we get:

   $$\lim_{x \to 3} \frac{f(x)}{g(x)} = \frac{-3}{1} = -3$$

2. From the Sum Law, we know that:

   $$\lim_{x \to 2} [f(x) + 5g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} 5g(x)$$

   and applying the Constant Multiple Law, we see that:

   $$\lim_{x \to 2} [f(x) + 5g(x)] = \lim_{x \to 2} f(x) + 5 \lim_{x \to 2} g(x)$$

   Examining the graph of $f(x)$ and $g(x)$, we can determine that

   $$\lim_{x \to 2} f(x) = -2$$

and

$$\lim_{x/to2} g(x) = 0$$

Substituting these values, we get:

$$\lim_{x\to 2} [f(x) + 5g(x)] = -2 + 5 \cdot 0 = -2$$

3. From the quotient law, we see that:

$$\lim_{x\to -1} \frac{3g(x)}{f(x)} = \frac{\lim_{x\to -1} 3g(x)}{\lim_{x\to -1} f(x)}$$

Applying the Constant Multiple Law, we get:

$$\lim_{x\to -1} \left[\frac{3g(x)}{f(x)}\right] = \frac{3\lim_{x\to -1} g(x)}{\lim_{x\to -1} f(x)}$$

From the graph, we see that:

$$\lim_{x\to -1} f(x) = 2$$

and

$$\lim_{x\to -1} g(x) = 2$$

Substituting, we get:

$$\lim_{x\to -1} \left[\frac{3g(x)}{f(x)}\right] = \frac{3 \cdot 2}{2} = 3$$

4. Applying the Product and Constant Multiple Laws, we get:

$$\lim_{x\to 0} [f(x) \cdot 5g(x)] = \lim_{x\to 0} f(x) \cdot 5 \cdot \lim_{x\to 0} g(x)$$

Examining the graphs, we see that $\lim_{x\to 0} f(x)$ does not exist and $\lim_{x\to 0} g(x) = 1$. Because $\lim_{x\to 0} f(x)$ does not exist, $\lim_{x\to 0} f(x) \cdot 5 \cdot \lim_{x\to 0} g(x)$ also does not exist.

5. Applying the Difference and Constant Multiple Laws, we see that:

$$\lim_{x\to -1} [f(x) - 3g(x)] = \lim_{x\to -1} f(x) - 3 \cdot \lim_{x\to -1} g(x)$$

Examining the graphs, we see that:

$$\lim_{x\to -1} f(x) = 2$$

and

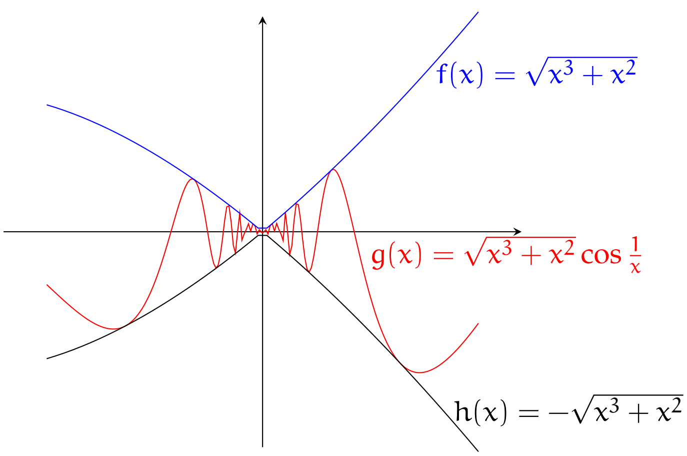
$$\lim_{x \to -1} g(x) = 2$$

Substituting, we get that:

$$\lim_{x \to -1} [f(x) - 3g(x)] = 2 - 3 \cdot 2 = 2 - 6 = -4$$

Answer to Exercise 9 (on page 20)

Let $f(x) = -\sqrt{x^3 + x^2}$ and $h(x) = \sqrt{x^3 + x^2}$. Near 0, $f(x) \leq \sqrt{x^3 + x^2} \cos \frac{1}{x} \leq h(x)$. Additionally, $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$. Therefore, by the Squeeze Theorem, we can state that $\lim_{x \to 0} \sqrt{x^3 + x^2} \cos \frac{1}{x} = 0$. Plotting all three functions, we can confirm our answer:



Answer to Exercise 10 (on page 21)

The question tells us that the function in question, $f(x)$, is between two other functions. $\lim_{x \to 2} 2x + 3 = 2(2) + 3 = 7$ and $\lim_{x \to 2} x^2 - 2x + 7 = 2^2 - 2(2) + 7 = 7$. Since the limits are equal, by Squeeze Theorem we can also know that $\lim_{x \to 2} f(x) = 7$.

Answer to Exercise 11 (on page 21)

Note that we can only evaluate the limit from the right as the domain for this function is

$x \geq 0$. Since the range of the sine function is $[-1, 1]$, we can state that

$$-1 \leq \sin \frac{\pi}{x} \leq 1$$

and therefore

$$\frac{1}{e} \leq e^{\sin \frac{\pi}{x}} \leq e$$

and because we assume the positive root, it is also true that

$$\frac{\sqrt{x}}{e} \leq \sqrt{x} e^{\sin \frac{\pi}{x}} \leq \sqrt{x} e$$

Taking the limits of the border functions, we see that

$$\lim_{x \to 0^+} \frac{\sqrt{x}}{e} = \lim_{x \to 0^+} \sqrt{x} e = 0$$

Therefore,

$$\lim_{x \to 0^+} \sqrt{x} e^{\sin \frac{\pi}{x}} = 0$$

Answer to Exercise 12 (on page 24)

1. define $f(x) = 2x^4 + x - 12$, $a = 1$, $b = 2$, and $N = 0$. Calculate $f(a)$ and $f(b)$:

   $$f(1) = -9 \text{ and } f(2) = 22$$

   Since $f(x)$ is a polynomial, it is continuous on the interval $x \in [1, 2]$ and we see that $f(a) < 0 < f(b)$. Therefore, there exists some $c \in [1, 2]$ such that $f(c) = 0$.

2. First, we can rearrange the equation we are considering and define $f(x) = \ln x - 3x + 4\sqrt{x}$ and realize we are looking for values where $f(x) = 0$. Both $\ln x$ and $\sqrt{x}$ are only continuous for $x > 0$. The interval we are interested in, $x \in [2, 3]$, is in the domain of continuity for both $\ln x$ and $\sqrt{x}$. Defining $a = 2$, $b = 3$, and $N = 0$, we find that $f(a) = 0.35 > N > -0.973 = f(b)$. Since $N \in [f(b), f(a)]$, there must exist some c such that $f(c) = N = 0$ and there is a solution to the equation $\ln x = 3x - 4\sqrt{x}$ on the interval $x \in (2, 3)$.

3. Similar to above, define $f(x) = 2 \sin x - 3x^2 + 2x$, $a = 1$, $b = 2$, and $N = 0$. Calculate that $f(a) = 0.683$ and $f(b) = -6.181$. Since $f(x)$ is continuous on the interval $x \in [1, 2]$ and $f(b) < N < f(a)$, there exists some $c \in (1, 2)$ such that $f(c) = 0$. Therefore, there is a solution to $\sin x = 3x^2 - 2x$ on the interval $x \in (1, 2)$.