Implicit Differentiation

Implicit differentiation is a technique in calculus for finding the derivative of a relation defined implicitly, that is, a relation between variables x and y that is not explicitly solved for one variable in terms of the other.

1.1 Implicit Differentiation Procedure

Consider an equation that defines a relationship between x and y:

$$F(x,y) = 0$$

To find the derivative of y with respect to x, we differentiate both sides of this equation with respect to x, treating y as an implicit function of x:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{F}(x,y) = \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{0}$$

Applying the chain rule during the differentiation on the left side of the equation gives:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Finally, we solve for $\frac{dy}{dx}$ to find the derivative of y with respect to x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

This result is obtained using the implicit differentiation method.

1.2 Example

Consider the equation of a circle with radius r:

$$x^2 + y^2 = r^2$$

First, we'll find $\frac{dy}{dx}$ the without implicit differentiation. Then, we'll apply implicit differentiation to get the same result.

1.2.1 Without Implicit Differentiation

First, we need to re-arrange the equation to solve for y:

$$y^{2} = r^{2} - x^{2}$$
$$y = \pm \sqrt{r^{2} - x^{2}}$$

We take the derivative of y by applying the Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2 \pm \sqrt{r^2 - x^2}} \cdot (-2x) = \frac{-x}{\pm \sqrt{r^2 - x^2}}$$

Notice the denominator of this fraction is the same as the solution we found for y, $y = \pm \sqrt{r^2 - x^2}$. So we can also represent this as:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y}$$

1.2.2 With Implicit Differentiation

With implicit differentiation, we assume y is a function of x and apply the Chain Rule.

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^2 + y^2] = \frac{\mathrm{d}}{\mathrm{d}x}[r^2]$$

For x^2 and r^2 , we take the derivative as we normally would. For y^2 , we apply the Chain Rule as outlined above.

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y}$$

which is the same result as we found without implicit differentiation.

1.3 Folium of Descartes

It was relatively easy to rearrange the equation for a circle to solve for y, but that is not always the case. Consider the equation for the folium of Descartes (yes, that Descartes!):

$$x^3 + y^3 = 3xy$$

It is much more difficult to isolate y in this equation. In fact, were we to do so, we would need 3 separate equations to completely describe the original equation.

1.3.1 Example: Tangent to Folium of Descartes

In this example, we will use implicit differentiation to easily find the tangent line at a point on the folium.

(a) Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

(b) Find the tangent to the folium $x^3 + y^3 = 3xy$ at the point (2,2)

(c) Is there any place in the first quadrant where the tangent line is horizontal? If so, state the point(s).

Solution:

(a)
$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$$

 $3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$
 $x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$

Rearranging to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$
$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

(b) We already have the coordinate point, (2, 2), so to write an equation for the tangent line all we need is the slope. Substituting x = 2 and y = 2 into our result from part (a):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2-2^2}{2^2-2} = \frac{-2}{2} = -1$$

This is the slope, m. Using the point-slope form of a line, our tangent line is y-2 = -(x-2).

(c) Recall that in the first quadrant, x > 0 and y > 0. We will set our solution for $\frac{dy}{dx}$ equal to 0:

$$\frac{y-x^2}{y+2-x} = 0$$

which implies that

$$y - x^2 = 0$$

Substituting $y = x^2$ into the original equation:

$$x^{3} + (x^{2})^{3} = 3(x)(x^{2})$$

 $x^{3} + x^{6} = 3x^{3}$

Which simplifies to

$$x^6 = 2x^3$$

Since we have excluded x = 0 by restricting our search to the first quadrant, we can divide both sides by x^3 :

$$x^{3} = 2$$
$$x = \sqrt[3]{2} \approx 1.26$$

Substituting $x \approx 1.26$ into our equation for y:

$$y\approx 1.26^2=1.59$$

Therefore, the folium has a horizontal tangent line at the point (1.26, 1.59).

1.4 Practice

Exercise 1

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] If $\arcsin x = \ln y$, what is $\frac{dy}{dx}$?

<i>— Working Space</i>	

_ Answer on Page 7 _

Exercise 2

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC Exam.] The points (-1, -1) and (1, -5) are on the graph of a function y = f(x) that satisfies the differential equation $\frac{dy}{dx} = x^2+y$. Use implicit differentiation to find $\frac{d^2y}{dx^2}$. Determine if each point is a local minimum, local maximum, or inflection point by substituting the x and y values of the coordinates into $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

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Answers to Exercises

Answer to Exercise 1 (on page 4)

Using implicit differentiation, we see that:

$$\frac{d}{dx} \arcsin x = \frac{d}{dx} \ln y$$
$$\frac{1}{\sqrt{1 - x^2}} = \frac{1}{y} \frac{dy}{dx}$$

Multiplying both sides by y to isolate $\frac{dy}{dx}$, we find that:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\sqrt{1-x^2}}$$

Answer to Exercise 2 (on page 5)

First, we need to find $\frac{d^y}{dx^2}$: $\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}y$ $= 2x + \frac{dy}{dx} = 2x + x^2 + y$

At (-1, -1), $\frac{dy}{dx} = (-1)^2 + (-1) = 0$ and $\frac{d^2y}{dx^2} = 2(-1) + (-1)^2 + (-1) = -2 < 0$. Since the slope of y is zero and the graph of y is concave down, (-1, -1) is a local maximum. At (1, -5), $\frac{dy}{dx} = 1^2 + -5 = -4 \neq 0$ and $\frac{d^2y}{dx^2} = 2(1) + 1^2 + (-5) = -2 \neq 0$. Since neither the first nor second derivative of y are zero, (1, -5) is neither a local extrema nor an inflection point.





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