Chapter 1

Determinants and Inverse Matrices

1.1 Determinants

Checking the independence of multitudes of vectors may take an immense amount of time. What if you had a list of 5, 10, or even 100 vectors? The determinant of a matrix is a scalar value that also indicates whether the columns of a matrix are linearly independent. So, if you put all your vectors together in a matrix and take the determinant of that matrix, the result will tell you if all the vectors are independent or not. For a 2D matrix, the determinant is the area of the parallelogram defined by the column vectors. For a 3D matrix, the determinant is the volume of the parallelepiped (a six-dimensional figure formed by six parallelograms, such as a cube).[1]

Let's plot the parallelogram for this matrix (see figure 1.1):

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

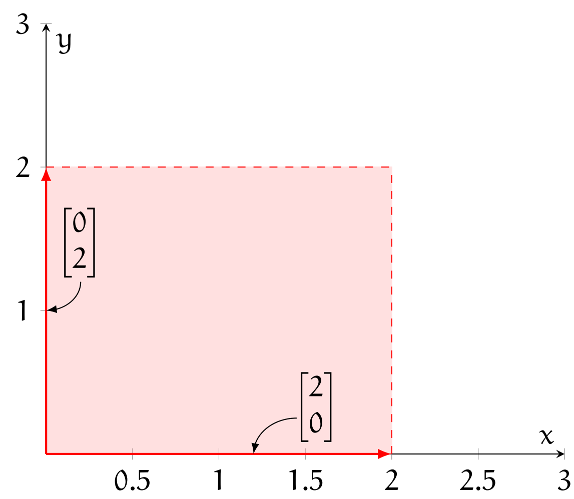


Figure 1.1: A parallelogram constructed from vectors $[2, 0]$ and $[0, 2]$

[1]Note that determinants can only be found for square, $n \times n$ matrices.

2 by 2 Determinant

The formal definition for calculating the determinant of a 2 by 2 matrix A is:

$$\det(A) = (a \cdot d) - (b \cdot c)$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For the matrix plotted above, the determinant is $(2 * 2) - (0 * 0)$. You can also see that 4.0 is the area, base (2) times height (2).

You can use the determinant to see what happens to a shape when it goes through a linear transformation. Let's scale the 2 by 2 matrix by 4:

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

Plot it (see figure 1.2):

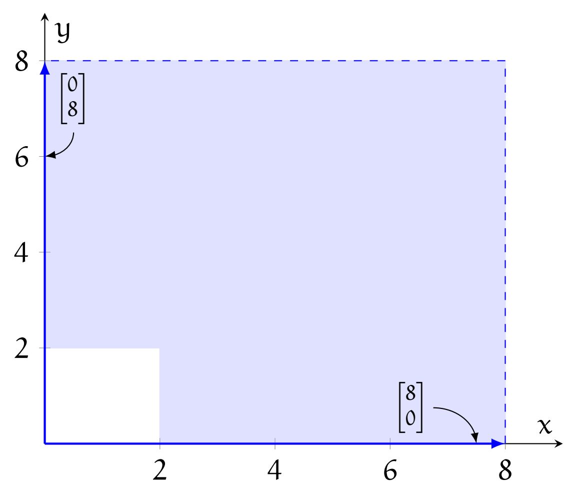


Figure 1.2: Scaling the matrix also scales the parallelogram.

Find the determinant using $(8 * 8) - (0 * 0) = 64$

You can see that scaling the matrix scaled the area by the scaling factor squared (see figure 1.3).

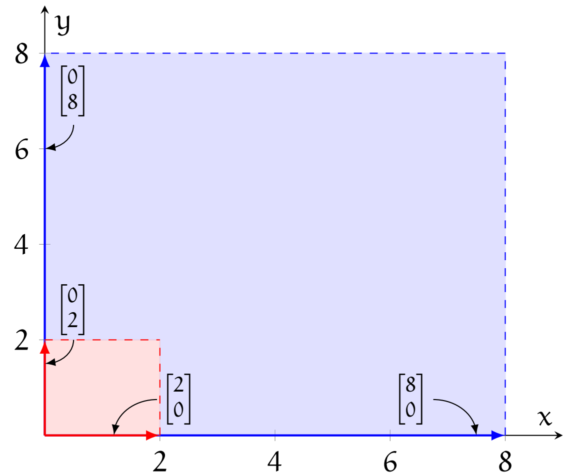


Figure 1.3: Scaling a matrix by a constant c increases the area of the parallelogram by a factor of c^2.

We can show why this is true mathematically. Suppose we have a 2 by 2 matrix A:

$$A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Then $\det(A) = wz - xy$. We can scale this matrix by a constant, c:

$$cA = c \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} cw & cx \\ cy & cz \end{bmatrix}$$

And we can take the determinant:

$$\det(cA) = \det\left(\begin{bmatrix} cw & cx \\ cy & cz \end{bmatrix}\right) = cw(cz) - cx(cy) = c^2(wz - xy) = c^2 \cdot \det(A)$$

Therefore, scaling a 2 by 2 matrix by a factor changes the determinant by that factor squared. What about higher dimensions? If each side of a cube were scaled by a factor of c, then the volume of the cube would change by a factor of c^3 (feel free to confirm this yourself). And if a tesseract (a four-dimensional cube) had each side scaled by a factor of c, then the hypervolume (four-dimensional volume) would be scaled by a factor of c^4. Do you notice a pattern?

In fact, scaling an $n \times n$ matrix by a constant factor, c, changes the determinant of that $n \times n$ matrix by a factor of c^n.

What happens if the columns of a matrix are not independent? Let's plot this matrix (see figure 1.4):

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

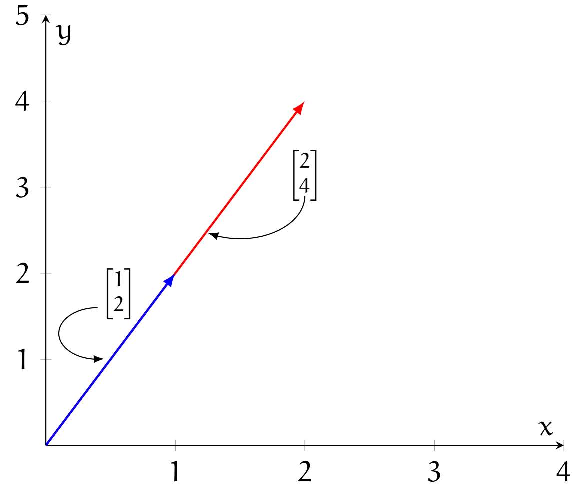


Figure 1.4: The vectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$ are co-linear, so there is no area between them and the determinant of $\begin{bmatrix}2 & 1\\4 & 2\end{bmatrix}$ is zero.

One vector overwrites the other. As you can see, the area is 0 because there is no space between the vectors. Therefore, the columns of the matrix are linearly dependent.

Exercise 1 Finding the Determinant

Plot the parallelogram represented by the columns of the matrix. What is the area of this parallelogram?

1. $\begin{bmatrix}1 & 4\\-3 & 1\end{bmatrix}$
2. $\begin{bmatrix}5 & -5\\5 & -1\end{bmatrix}$
3. $\begin{bmatrix}0 & -5\\-2 & 0\end{bmatrix}$

Working Space

Answer on Page 15

Calculating the determinant for a 2 by 2 matrix is easy. For a larger matrix, finding the determinant is more complex and requires breaking down the matrix into smaller matrices until you reach the 2x2 form. The process is called expansion by minors. For example,

3×3 Determinant

The determinant of a 3 by 3 matrix is found by

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

A trick for this is rewriting the matrix as a 3×5 augmented matrix with the first 2 columns after the third column:

$$B = \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

and solving the *down-right diagonals* minus the *down-left diagonals*:

$$\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$$

Down-right diagonals (add)   Down-left diagonals (subtract)

$$\det(B) = (a \cdot e \cdot i) + (b \cdot f \cdot g) + (c \cdot d \cdot h) - [(c \cdot e \cdot g) + (a \cdot f \cdot h) + (b \cdot d \cdot i)]$$

Note that this is the same multiplication as above, just formatted differently.

As you can see, this involves a recursive process of breaking a larger matrix into a smaller 2×2 matrix.

For our purposes, we simply want to first check to see if a matrix contains linearly independent rows and columns before using our Python code to solve.

1.2 Determinants in Python

Modify your code so that is uses the $np.linalg.det()$ function. If the determinant is not zero, then you can call the $np.linalg.solve()$ function. Your code should look like this:

```
# Are the rows and columns independent?
```

```
# Equivallently, is the determinant 0?
if (np.linalg.det(D) != 0):
    j = np.linalg.solve(D,e)
    print(j)
else:
    print("Rows and columns are dependent.")
```

How does this work below the hood? Let's also write a recursive python function that finds our determinant:

There are two base cases:

- The matrix is of size 1×1
- The matrix is of size 2×2

And further sizes can be simplified into one of the base cases by *cofactor expansion*. The idea behind cofactor expansion is to break a big determinant into smaller ones until we reach cases we already know how to solve. Formally, this is written as

$$|A| = \sum_{j=1}^{n} (-1)^{(i+j)} a_{(ij)} M_{(ij)}$$

We do this as follows:

1. Pick the first row of the matrix.
2. For each entry in that row:
   - Remove the row and column containing that entry.
   - This creates a smaller matrix, called a *minor*, which are submatrices. Recall what we did above for our 3×3 determinant definition.
3. Compute the determinant of that smaller matrix.
4. Multiply the original entry (where the row and column lines originate from), the determinant of its minor, and an alternating sign $+, -, +, \ldots$.
5. Sum all of these results together.

This process repeats recursively: each smaller determinant is computed the same way, until we reach one of the base cases.

If we format our matrices as a nested array, we can use python's indexing to check and reduce the matrices. Take a look at this recursive determinant program:

```
def recursive_determinant(matrix):
```

```
    """
    Compute the determinant of a square matrix recursively.
    matrix: list of lists, forming a matrix of size n by n
    """
    n = len(matrix)

    # base case 1x1:
    if n == 1:
        return matrix[0][0]
    # base case 2x2
    if n == 2:
        return matrix[0][0] * matrix[1][1] - matrix[0][1] * matrix[1][0]

    det = 0

    for col in range(n):
        # build a minor matrix:
        minor = []
        # for every row in the matrix
        for row in matrix[1:]:
            # remove column col and store as new_row
            new_row = row[:col]+row[col+1:]
            # append new row to the minor matrix
            minor.append(new_row)

        sign = (-1) ** col #exponential for alternating sign
        det += sign * matrix[0][col] * recursive_determinant(minor)
    return det
```

We implemented our known base case, and recursively reduce our array until it fits a known base case. This is exactly how numpy's `np.linalg.det()` works.

1.3 Inverse Matrices

Now that we have talked about determinants, we can talk about Inverse Matrices. The idea of a matrix inverse is a natural generalization of the multiplicative inverse of a real number. For example, since

$$3 \cdot \frac{1}{3} = 1,$$

the number $\frac{1}{3}$ is called the multiplicative inverse of 3.

Similarly, for matrices, we define an inverse in terms of matrix multiplication and the identity matrix. If A is an $n \times n$ matrix, its inverse (when it exists) is denoted by A^{-1} and satisfies

$$AA^{-1} = A^{-1}A = I_n,$$

where I_n is the $n \times n$ identity matrix.

Inverse Matrix

An inverse matrix is a square matrix that satisfies the following property:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

where A is the original matrix, B is the inverse matrix, and I_n is the identity matrix of size $n \times n$.

When such a matrix B exists, it is denoted by A^{-1}, and is said to be *invertible*. Note that sometimes an inverse matrix may not exist.

It is important to note that not every matrix has an inverse. In particular, only *square matrices* can be invertible, and even among square matrices, an inverse may fail to exist.

1.3.1 Existence of Square Matrices

Existence of the Inverse

A square matrix A has an inverse if and only if its determinant is nonzero:

$$A^{-1} \text{ exists if and only if } \det(A) \neq 0.$$

If

$$\det(A) = 0,$$

then A is called a **singular matrix**, and no inverse exists. Equivalently, an inverse matrix does not exist when the rows or columns of A are *linearly dependent*. As discussed in the chapter on linear dependence, this occurs when one row (or column) is a scalar multiple of another, or when a row (or column) can be written as a linear combination of the others.

Linear dependence implies that the matrix does not contain enough independent information to reverse its action, making an inverse impossible.

1.3.2 Finding an Inverse Matrix

2×2 Inverse

There are several methods for finding the inverse of a matrix. For small matrices, especially 2×2 matrices, there is a direct formula. For larger matrices, a more systematic method using row operations is required.

Inverse of a 2×2 Matrix

A trick for a 2×2 Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that we multiply by the a new matrix has a and d swapped, and b and c switch signs. Finally, we divide by the determinant of the original A.

This formula is valid only when the determinant of A is nonzero. Recall that the determinant of a 2×2 matrix is

$$\det(A) = ad - bc.$$

If $ad - bc = 0$, then A is singular and has no inverse.

Exercise 2 **2×2 Practice**

Find the inverse of the matrix of

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

Working Space

Answer on Page 16

$n \times n$ Inverse

For matrices of size 3×3 or greater, we create an augmented matrix.

Recall that we can create an augmented matrix by writing two matrices directly next to each other.

To find the inverse of a square matrix A:

1. Form the augmented matrix $[A \mid I_n]$, where I_n is the identity matrix.
2. Use elementary row operations to transform the left side into I_n.
3. If this is possible, the right side of the augmented matrix becomes A^{-1}.

Symbolically, this process can be written as

$$[A \mid I_n] \longrightarrow [I_n \mid A^{-1}].$$

If row reduction does not result in the identity matrix on the left side, then A does not have an inverse. In this case, the matrix is singular.

Exercise 3 A bigger inverse

Working Space

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & -3 \end{bmatrix}$$

Check your answer using $AA^{-1} = I_3$

Answer on Page 16

Exercise 4 Does a matrix exist?

Working Space

Given the matrix:

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 8 & 16 \\ 4 & 8 & 16 & 32 \\ 8 & 16 & 32 & 64 \end{bmatrix}$$

Does an inverse matrix exist? Explain why or why not

Answer on Page 18

The augmented matrix row-reduction method not only provides a way to compute inverses, but also offers another criterion for invertibility: a matrix is invertible if and only if it can be row-reduced to the identity matrix.

1.3.3 Relation to $A\vec{x} = \vec{b}$

An inverse matrix is not only for square matrices, but also for systems of equations and our fundamental linear algebra equation $A\vec{x} = \vec{b}$. Let's look at an example.

Given

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

solve for $\vec{x}$ by first finding the inverse.

First, we can interpret our givens as a systems of matrices:

$$A\vec{x} = \vec{b} \quad \Longleftrightarrow \quad \begin{cases} 2x_1 + x_2 = 5 \\ x_1 + x_2 = 3 \end{cases}$$

Checking that an inverse matrix first exists, we have

$$\det(A) = 2(1) - 1(1) = 1 \neq 0$$

So we know an inverse matrix exists! To find A^{-1}

$$A^{-1} = 1 \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Here is the fun part! Let's do some computational equivalences:

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}(A\vec{x}) &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned} \tag{1.1}$$

Equation (1.1) show that, by only multiplying by the matrix equivalent of 1, the identity matrix I, we can state that $\vec{x} = A^{-1}\vec{b}$. Since we have both A^{-1} and $\vec{b}$, we can find $\vec{x}$:

$$x = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Geometrically, this did a very simple operation. We used the fact that A vectors transforms $\vec{x}$ into $\vec{b}$. Then we noted that A^{-1} undoes that transformation, recovering our original $\vec{x}$ by finding $A^{-1}\vec{b}$. This may be hard to visualize, so don't worry if it is hard to grasp immediately. We will review it again in a following chapter.

1.3.4 Matrix Inverse using NumPy

We can use our Python library NumPy to find inverses a lot easier using the `np.linalg.inv` function to find the inverse of a given array.

```
import numpy as np

A = np.array([[2, 1],
              [1, 1]])

A_inv = np.linalg.inv(A)
print(A_inv)
```

This outputs

```
[[ 1. -1.]
 [-1.  2.]]
```

If given a matrix that is *linearly dependent*, NumPy will raise an exception:

```
raise LinAlgError("Singular matrix")
numpy.linalg.LinAlgError: Singular matrix
```

letting us know that the inverse does not exist.

To improve our code, we check for the determinant of the matrix beforehand. In this code, we introduce a tolerance as computers handle numbers near zero as very small but non-zero digits.

```
import numpy as np

A = np.array([[2, 1],
              [4, 2]], dtype=float)

# BAD PRACTICE:
# A_inv = np.linalg.inv(A)
# print(A_inv)

detA = np.linalg.det(A)

tolerance = 1e-12

if abs(detA) < tolerance:
    print("Matrix is singular or nearly singular. No reliable inverse.")
else:
    A_inv = np.linalg.inv(A)
    print("Determinant:", detA) # you may see a large floating point number
    print("Inverse:\n", A_inv)
```

Output:

```
Matrix is singular or nearly singular. No reliable inverse.
```

1.4 Summary

In this chapter, we established the determinant of a matrix and inverse matrix.

The determinant of a matrix provides a powerful geometric and algebraic description of how a matrix acts on space.

In two dimensions, the determinant of a 2×2 matrix represents the signed area of the parallelogram formed by the images of the standard basis vectors. In higher dimensions, the determinant represents signed volume.

Further, if a matrix A multiplies a region in space, then $|\det(A)|$ describes the factor by which areas or volumes are scaled, noting the sign and absolute value of the determinant as a scalar.

The inverse of a matrix *reverses the effect of the original matrix*. If A is invertible, then multiplying by A^{-1} restores any vector to its original position:

$$AA^{-1} = A^{-1}A = I.$$

Thus, invertibility, nonzero determinant, linear independence of rows and columns, and reversibility of action are all different ways of describing the same underlying property.

In the next chapter, we will interpret matrices as *functions* that transform vectors.

This idea will allow us to visualize matrix multiplication, understand invertibility deeper, and connect algebraic properties such as determinants to transformations of space.

The existence of an inverse matrix is closely related to whether a matrix represents a reversible transformation.

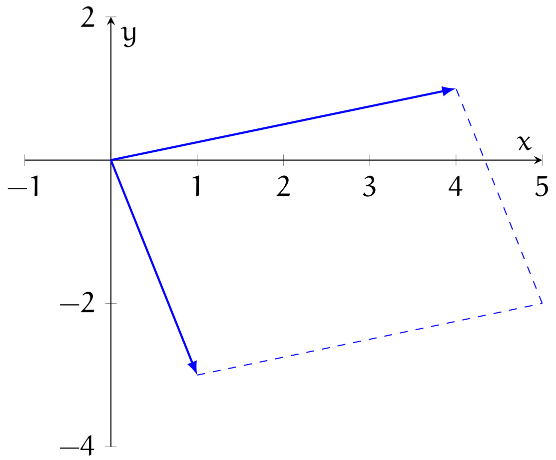
The next chapter will be very graph and program heavy, as we expand the geometric properties of determinants, inverses, and matrices.

APPENDIX A

Answers to Exercises

Answer to Exercise 1 (on page 4)

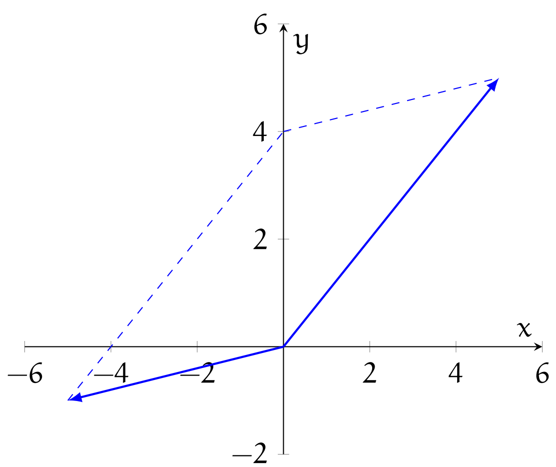
1. Our two vectors from the columns of the matrix are $[1, -3]$ and $[4, 1]$. Plotting:



The area of this parallelogram is the same as the determinant of the matrix:

$$\det\left(\begin{bmatrix} 1 & 4 \\ -3 & 1 \end{bmatrix}\right) = 1 \cdot 1 - (4 \cdot -3) = 1 + 12 = 13$$

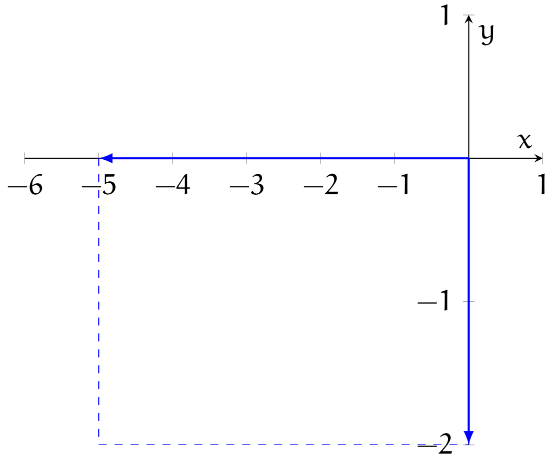
2. Our two vectors from the columns of the matrix are $[5, 5]$ and $[-5, -1]$. Plotting:



The area of this parallelogram is the same as the determinant of the matrix:

$$\det\left(\begin{bmatrix}5 & -5\\ 5 & -1\end{bmatrix}\right) = 5 \cdot -1 - (-5 \cdot 5) = -5 + 25 = 20$$

3. Our two vectors from the columns of the matrix are $[0, -2]$ and $[-5, 0]$. Plotting:



This is a rectangle, and we can see the area is $5 \cdot 2 = 10$. However, the determinant is:

$$\det\left(\begin{bmatrix}0 & -5\\ -2 & 0\end{bmatrix}\right) = 0 \cdot 0 - (-5 \cdot -2) = 0 - 10 = -10$$

We will discuss this unusual response in a future chapter.

Answer to Exercise 2 (on page 9)

First, find the determinant of A:

$$\det(A) = (1)(-1) - (2)(0) = -1$$

So our inverse is given by:

$$A^{-1} = -1\begin{bmatrix}-1 & -2\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}$$

Answer to Exercise 3 (on page 10)

First, check that A has an inverse:

$$\det A = 1\begin{vmatrix}1 & 1\\ 4 & -3\end{vmatrix} - 2\begin{vmatrix}0 & 1\\ 2 & -3\end{vmatrix} - 2\begin{vmatrix}0 & 1\\ 2 & 4\end{vmatrix} = 1(-3-4) - 2(0-2) - 2(0-2) = -7 + 4 + 4 = 1$$

Since $\det(A) \neq 0$, A must have an inverse. We find it's inverse by the augmented matrix method:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -3 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & -2 & 4 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right].$$

This gives us our inverse on the right side, $A^{-1} = \begin{bmatrix} -7 & -2 & 4 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix}$

To check our answer, we can use $AA^{-1} = I_3$:

$$\begin{aligned}
AA^{-1} &= \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & -3 \end{bmatrix} \begin{bmatrix} -7 & -2 & 4 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1(-7)+2(2)+(-2)(-2) & 1(-2)+2(1)+(-2)(0) & 1(4)+2(-1)+(-2)(1) \\ 0(-7)+1(2)+1(-2) & 0(-2)+1(1)+1(0) & 0(4)+1(-1)+1(1) \\ 2(-7)+4(2)+(-3)(-2) & 2(-2)+4(1)+(-3)(0) & 2(4)+4(-1)+(-3)(1) \end{bmatrix} \\
&= \begin{bmatrix} -7+4+4 & -2+2+0 & 4-2-2 \\ 0+2-2 & 0+1+0 & 0-1+1 \\ -14+8+6 & -4+4+0 & 8-4-3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.
\end{aligned}$$

Answer to Exercise 4 (on page 11)

No. Each row is a scalar multiple of the first row:

- Row 2 = 2 × Row 1
- Row 3 = 4 × Row 1
- Row 4 = 8 × Row 1

So the rows are *linearly dependent*, meaning the matrix has rank 1, and its determinant is 0. A matrix with determinant 0 is singular, so no inverse exists.

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