Chapter 1

Linear Independence and Dependence

We have briefly mentioned the idea of linear independence and dependence in previous chapters. Let's explicitly define them using the concepts we have covered already.

1.1 Linear Independence

Linear Independence

Linear independent vectors are vectors that cannot be written as linear combinations (scalar multiples) of each other.

Mathematically, this means that

$$c_1v_1 + c_2v_2 \dots c_kv_k = 0$$

is only satified by

$$c_1 = c_2 = \cdots = c_k = 0$$

Linear Dependence

A set of vectors $\{v_1, \dots, v_k\}$ is **linearly dependent** if there exist scalars $c_1, \dots, c_k$, not all zero, such that

$$c_1v_1 + \cdots + c_kv_k = \mathbf{0}.$$

For two vectors, this reduces to one being a scalar multiple of the other.

When vectors are arranged as columns of a matrix, linear independence means that no column can be written as a linear combination of the others.

We say that each vector contributes a new *direction* not captured by any previous ones, meaning that no vector can be written as a linear combination of the others. Each vector adds something genuinely new to the set, so there is no redundancy.

Exercise 1 Testing Linear Independence

Determine whether each pair of vectors is linearly independent or linearly dependent.

1. $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$
2. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ -2 \end{bmatrix}$
3. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
4. $\begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$

Working Space

Answer on Page 9

Any set of vectors that includes the zero vector is automatically linearly dependent, since

$$1 \cdot \vec{0} = \vec{0}$$

gives a nontrivial solution.

Exercise 2 Zero Vector and Dependence

Determine whether each set of vectors is linearly independent.

1. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
2. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$
3. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Working Space

Answer on Page 9

Geometrically, in $\mathbb{R}^2$: independent vectors are not collinear. In $\mathbb{R}^2$, the vectors are generally perpendicular.

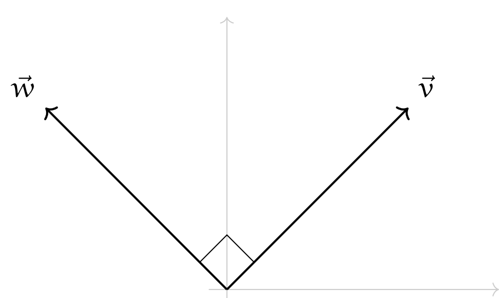

Figure 1.1: An example of independent vectors in $\mathbb{R}^2$.

In $\mathbb{R}^3$, two independent vectors lie in a plane, while three independent vectors do not lie in the same plane.

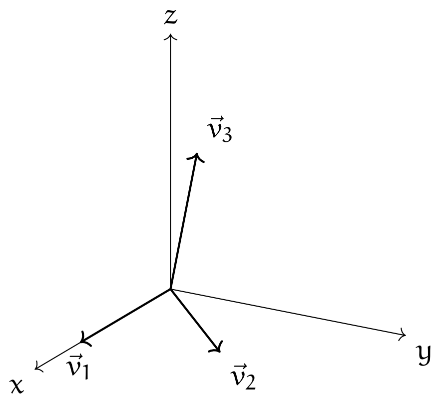

Figure 1.2: An example of linearly independent vectors in $\mathbb{R}^3$.

Exercise 3 Values Causing Dependence

Find all values of k for which the vectors are linearly dependent.

1. $\begin{bmatrix} 1 \\ k \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
2. $\begin{bmatrix} k \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2k \\ 2 \\ 4 \end{bmatrix}$

Working Space

Answer on Page 9

1.1.1 Connection to Span

The *span* of a set of vectors is the collection of all vectors that can be written as linear combinations of those vectors. In other words, the span consists of all vectors that are reachable using the given set.

We now use this idea to better understand linear independence.

Consider the vector space $\mathbb{R}^3$. Let

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Any vector in $\mathbb{R}^3$ can be written as a linear combination of these three vectors. Therefore,

$$\text{span}(V) = \mathbb{R}^3.$$

Moreover, no vector in V can be written as a linear combination of the other two. Each vector contributes a new direction to the span. For this reason, the vectors in V are *linearly independent*.

Equivalently, a set of vectors is linearly independent if every vector in their span has a *unique* representation as a linear combination of those vectors.

This example illustrates an important idea: a linearly independent set contains no redundant vectors, and when such a set spans a space, the set is called a basis for the space.

A set B is a basis if its elements are linearly independent and every element of V is a linear combination of elements of B. In other words, a basis is a linearly independent spanning set. A vector space can have several bases; however all the bases have the same number of elements, called the dimension of the vector space. We will dive into this concept more in the subspaces chapter, so don't worry if this is a bit of a jump!

1.2 Linearly Dependent Vectors: Geometrically

Two vectors are linearly dependent if one is a multiple of the other. Mathematically,

Linearly dependent vectors in $\mathbb{R}^n$

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are linearly dependent if there exists a scalar $a \in \mathbb{R}$ such that

$$\mathbf{v} = a\,\mathbf{u}.$$

Graphically, two linearly dependent vectors in $\mathbb{R}^2$ lie on the same line through the origin (or one of them is the zero vector).

If two vectors are linearly dependent, then linear combinations of them can only produce vectors lying on that same line. If they are *not* linearly dependent, they are called linearly *independent*, and their linear combinations can produce every vector in $\mathbb{R}^2$.

Example: Which of the following 3 vectors are linearly dependent, if any? $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 6 \\ -8 \\ 2 \end{bmatrix}$.

Solution: Two vectors are linearly dependent if one is a scalar multiple of the other. Let's compare **u** and **v**. Since the first component of **u** is 1 and the first component of **v** is -3, let's multiply **u** by -3 to see if we get **v**:

$$-3\mathbf{u} = -3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix} \neq \mathbf{v}$$

Therefore, **u** and **v** are *not* linearly dependent. Now let's examine **v** and **w**. Again, we will use the first components: the first component of **w** is 6, so let's see if multiplying **v** by -2 yields **w**:

$$-2\mathbf{v} = -2 \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 2 \end{bmatrix} = \mathbf{w}$$

Therefore, **v** and **w** are linearly dependent. Since we already know that **u** and **v** are not linearly dependent, we also know that **u** and **w** are also not linearly dependent.

Exercise 4 Linear Dependence

Identify which, if any, of the following vectors are linearly dependent:

1. $\mathbf{a} = \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}$

2. $\mathbf{b} = \begin{bmatrix} -4 \\ 5 \\ -3 \end{bmatrix}$

3. $\mathbf{c} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

4. $\mathbf{d} = \begin{bmatrix} 1 \\ -\frac{1}{4} \\ -1 \end{bmatrix}$

5. $\mathbf{e} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

6. $\mathbf{f} = \begin{bmatrix} -6 \\ \frac{3}{2} \\ 6 \end{bmatrix}$

Working Space

Answer on Page 10

1.3 What's next?

In this chapter, we defined linear independence and linear dependence, which describe whether a set of vectors contains redundancy. Linearly independent vectors introduce new directions, while dependent vectors can be written as linear combinations of others.

These ideas lead naturally to the study of subspaces, which describe collections of vectors that are closed under addition and scalar multiplication. In the next section, we formalize how sets of vectors generate larger structures and how linear independence helps describe them efficiently.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 2)

Two vectors are linearly dependent if one is a scalar multiple of the other.

1. Dependent, since $\begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
2. Dependent, since $\begin{bmatrix} -6 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
3. Independent, since neither vector is a scalar multiple of the other.
4. Dependent, since $\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$.

Answer to Exercise 2 (on page 3)

Any set containing the zero vector is linearly dependent, since a nontrivial linear combination can equal the zero vector.

1. Dependent.
2. Dependent.
3. Dependent.

Answer to Exercise 3 (on page 4)

1. The vectors are dependent when $\begin{bmatrix} 1 \\ k \end{bmatrix}$ is a scalar multiple of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. This occurs when $k = 2$.

2. The vectors are dependent for all values of k, since

$$\begin{bmatrix} 2k \\ 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} k \\ 1 \\ 2 \end{bmatrix}.$$

Answer to Exercise 4 (on page 7)

We see that $\frac{\mathbf{a}}{-4} = -\frac{1}{4} \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{4} \\ -1 \end{bmatrix} = \mathbf{d}$. Additionally, $\frac{3}{2}\mathbf{a} = \frac{3}{2} \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ \frac{3}{2} \\ 6 \end{bmatrix} = \mathbf{f}$. Therefore, vectors **a**, **d**, and **f** are linearly dependent.

We also see that $\frac{1}{2}\mathbf{c} = \frac{1}{2} \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \mathbf{e}$. Therefore, vectors **c** and **e** are linearly dependent. Vector **b** is not linearly dependent to any of the other vectors.

Index