Chapter 1

Arc Lengths

1.1 Determining the Arc Length of a Curve

Another application of integrals is finding the length of a curve. In real-life, we could do this by laying a piece of string up against the curve, then straightening out the string and measuring its length with a ruler (you may have done this is elementary school when you were first learning about the relationship between the radius and circumference of a circle). Archimedes estimated the circumference of a circle by inscribing a circle with polygons of increasing numbers of sides. (Archimedes' proof that π is between $3\frac{10}{71}$ and $3\frac{1}{7}$ is more complicated but we won't dive into that here.) As we increase the number of sides of the inscribed polygon, the perimeter of the polygon (the sum of the lengths of the sides) gets closer to the circumference of the circle (see figure 1.1). Now, it's easy to find the length of a polygon: just add up the length of the line segments! Using this, we can find the length of a curve by approximating it as many short lines and adding up the lengths of those lines.

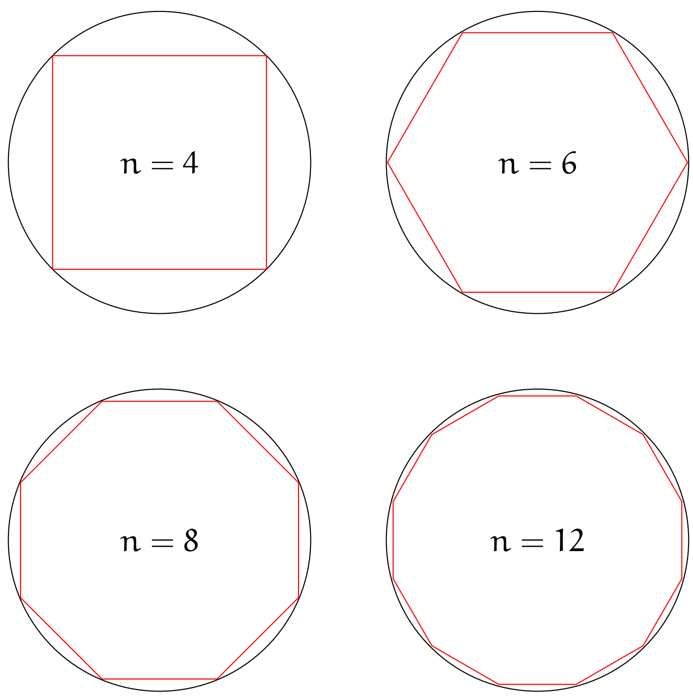

Figure 1.1: As n increases, the perimeter of the inscribed polygon approaches the circumference of the circle

We can choose n points along the graph of $f(x)$ and connect each point with a straight line (this is shown in figure 1.2). If we add up the length of the lines, we get an estimate of the length of the curve. We represent the length of the line between the i^{th} point, P_i and the previous point, P_{i-1} as $|P_{i-1}P_i|$ (recall that the absolute value sign can be used to signify the length of something). Therefore, the sum of the lengths of the lines approximating the curve is:

$$\sum_{i=1}^{n} |P_{i-1}P_i|$$

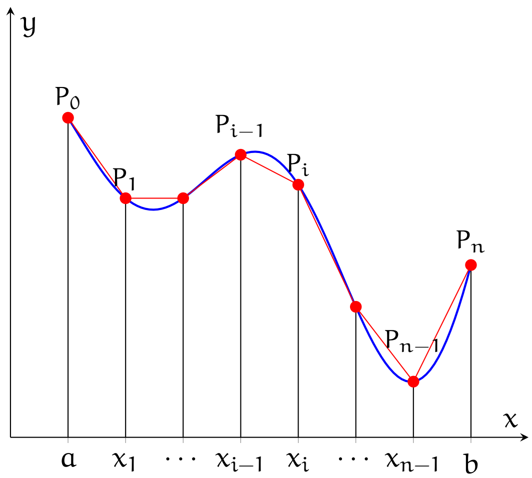

Figure 1.2: Polygon approximation of $f(x)$

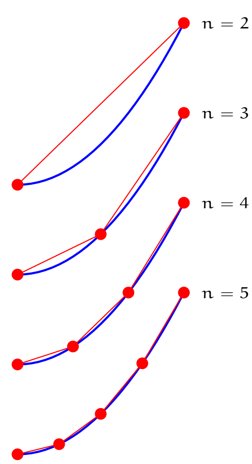

Figure 1.3: As the number of points increases, the total length of the lines segments approaches the true length of the curve

The more points we choose, the closer the lines lay to the actual curve (see figure 1.3), and the closer our estimate is to the true length. So, to find the true length, we will want to take n to ∞. Therefore, the actual curve length is the limit as $n \to \infty$ of that sum:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The length of each segment can be found using the Pythgorean theorem. Recall that the distance between two points on the xy-plane is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. (For a reminder of why this is, see figure 1.4.) The coordinates of P_{i-1} are $(x_{i-1}, f(x_{i-1}))$ and the coordinates of P_i are $(x_i, f(x_i))$. Substituting this into the above sum, we see that the total length of the segments is

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

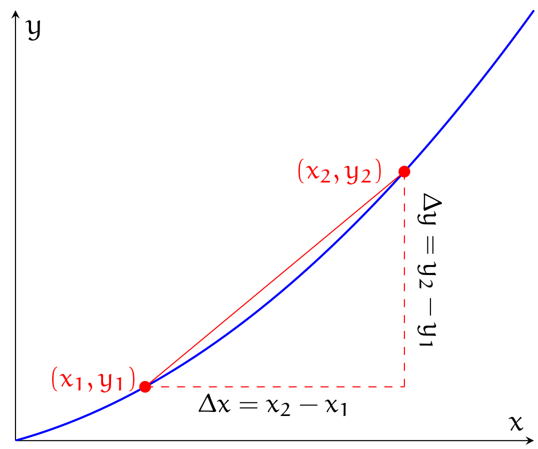

Figure 1.4: The distance between two points on the xy-plane

Recall the Mean Value Theorem, which states that there is some x_i^* such that $f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$. Substituting this into the above sum, we get:

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f'(x_i^*)(x_i - x_{i-1}))^2}$$

Recall from the chapter on Riemann sums and the integral that we defined $\Delta x = x_i - x_{i-1}$ and we can further re-write the sum as

$$\sum_{i=1}^{n} \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\sqrt{(\Delta x)^2} = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

Putting this all together, we see that that actual length of the curve is defined as

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

This is the definition of the integral of $\sqrt{1 + [f'(x)]^2}$ and therefore the length of some function $f(x)$ on the interval $a < x < b$ is $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. In another notation, this is equivalent to $\int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$.

1.2 Arc Length of Vector-valued Functions

Suppose you have a vector-valued function, $f(t) = [x(t), y(t)]$. A common example might be an artillery shell shot at an angle. For a shell shot with an initial velocity v_o at angle θ from the ground, its position can be described with the vector-valued function $f(t) = [v_o \cos\theta(t), v_o \sin\theta(t) - 4.9t^2]$. (A concrete example where $v_o = 12\frac{m}{s}$ and $\theta = 30^o$ is shown in figure 1.5.)

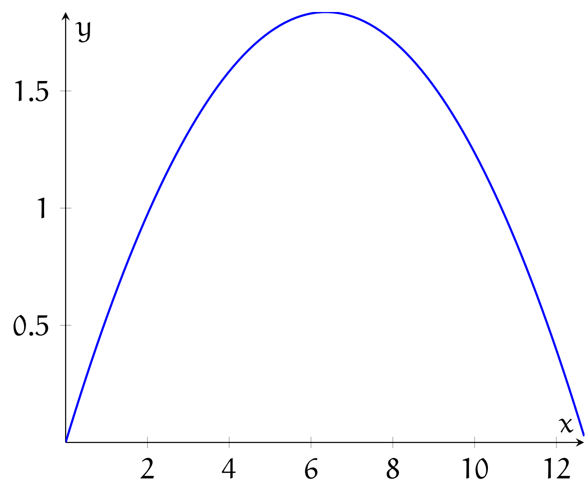

Figure 1.5: The path of an artillery shell with shot with initial velocity v_o at angle θ

How can we find the length of the flight path of the artillery shell? We can re-interpret the length integral for a vector-valued function, $f(t) = [x(t), y(t)]$.

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_a^b \sqrt{1 + \frac{(dy/dt)^2}{(dx/dt)^2}} \frac{dx}{dt}\, dt$$

Moving the $\frac{dx}{dt}$ under the square root, we see that

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$$

Which is equivalent to

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dx$$

1.3 Applications in Physics

When we take the integral of a velocity function, we get the *displacement*. For example, if you drove to school and home again, your displacement would be zero. However, the *distance* you traveled is not zero! We can use the arc length formula to find the total distance traveled. (Remember, that if $x(t)$ is the object's position, then its velocity is given by $x'(t)$.)

Suppose a block is attached to a spring on a frictionless horizontal surface. You pull on the block, initiating harmonic motion described by $v(t) = (-0.16)\sin 9t$, where v is in $\frac{m}{s}$ and t is in *sec*. (Note: we are working in radians, not degrees.) What is the block's displacement from $t = 0$ to $t = 3$? What is the total distance the block moves from $t = 0$ to $t = 3$?

To find the displacement, we integrate the velocity function over the specified interval:

$$\int_0^3 (-0.16)\sin 9t\,dt = \frac{-0.16}{9}(-\cos 9t)|_0^3$$

$$= \frac{0.16}{9}[\cos 27 - \cos 0] = \frac{(0.16)(-0.29-1)}{9} = -0.0229m$$

The position function and displacement are shown in figure 1.6. To find the total distance traveled, we need to find the length of the curve. Before we do so, take a minute to mentally predict: will the distance be more or less than the displacement?

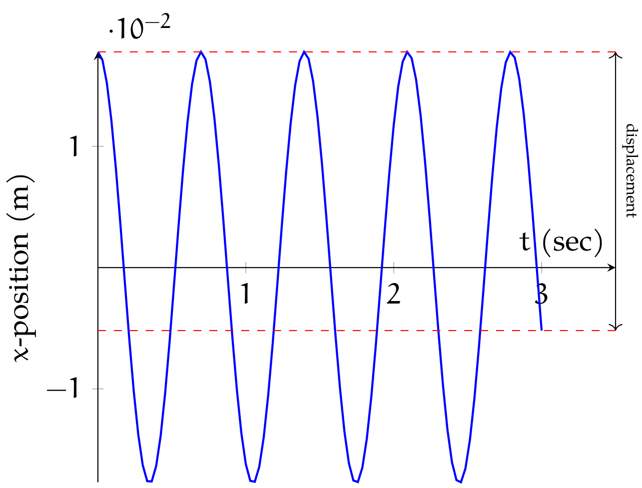

Figure 1.6: The position of the block with displacement shown. The distance traveled is the total length of the curve

Recalling that the distance traveled by an object is $\int_a^b \sqrt{1+[x'(t)]^2}\,dt = \int_a^b \sqrt{1+[v(t)]^2}\,dt$, we can write an integral to determine the total distance traveled by the block:

$$\int_0^3 \sqrt{1+[-0.16\sin 9t]^2}\,dt$$

Unfortunately, we do not know an antiderivative for this integral, and u-substitution won't help us. However, for definite integrals, calculators such as a TI-89 or Wolfram Alpha can easily use Riemann sums to determine the value of the integral to a high precision. Using such a tool, we find that the total distance traveled by the block is ≈ 3.019 meters. Did you predict that the distance would be greater than the displacement?

1.4 Practice

Exercise 1

Write an integral that gives the length of the requested curve.

1. $y = \ln x$ from $x = 1$ to $x = 3$
2. $y = \sin x$ from $x = 0$ to $x = \pi$
3. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 4$
4. $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{3}$

Working Space

Answer on Page 9

Exercise 2

The arc length function for a curve $f(x)$, where f is an increasing function, is given by $s(x) = \int_0^x \sqrt{3t+5}\,dt$. If f has y-intercept 2, find an equation for f. What point on f is three units from the y-intercept? Give your answer to the thousandths place.

Working Space

Answer on Page 9

Exercise 3

[This problem was originally presented as a no-calculator, multiple-choice question on the 2012 AP Calculus BC exam.] Write an integral that gives the length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

Working Space

Answer on Page 10

Exercise 4

An out of control rocket ship is spiraling out of control through space. Its velocity can be described with the vector-valued function $v(t) = [-1412 \sin t, 1412 \cos t, t]$ where v is in $\frac{m}{s}$ and t is in sec. How far does the ship travel in the first 60 seconds? In the second 60 seconds? [Hint: in three dimensions, the length of a vector-valued function is $\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt$]

Working Space

Answer on Page 10

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

1. $L = \int_1^3 \sqrt{1 + \frac{1}{x^2}}\,dx$
2. $L = \int_0^\pi \sqrt{1 + \cos^2 x}\,dx$
3. $L = \int_1^4 \sqrt{1 + (x^2 - \frac{1}{4x^2})^2}\,dx$
4. $L = \int_0^{\frac{\pi}{3}} \sqrt{1 + (\frac{1}{\cos x} \times -\sin x)^2}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\,dx$

Answer to Exercise 2 (on page 6)

From looking at the structure of the given arc length integral, we can see that $f'(t) = \sqrt{3t+4}$. Taking the antiderivative, we find that $f(x) = \frac{2}{9}(3x+4)^{3/2} + C$. Substituting $f(0) = 2$, we can solve for C.

$$\begin{aligned}
2 &= \frac{2}{9}(3(0)+4)^{3/2} + C \\
2 &= \frac{2}{9}(4)^{3/2} + C \\
2 &= \frac{2}{9}(2)^3 + C \\
2 &= \frac{16}{9} + C \\
\frac{18}{9} &= \frac{16+9C}{9} \\
18 &= 16 + 9C \\
2 &= 9C \\
C &= \frac{2}{9}
\end{aligned}$$

Therefore, $f(x) = \frac{2}{3}(3x+4)^{3/2} + \frac{2}{9}$. To find the coordinate point where $s(x) = 3$, we first note that the antiderivative of $\sqrt{3t+5}$ is $\frac{2}{9}(3t+5)^{3/2} + C$. Therefore, $s(x) = \frac{2}{9}(3x+5)^{3/2} - \frac{2}{9}(5)^{3/2}$. Setting $s(x) = 3$ and solving for x, we find that $x = 1.159$

Answer to Exercise 3 (on page 7)

Recall that arc length is given by $\int_a^b \sqrt{1+[f'(x)]^2}\,dx$. Since $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. Taking $a = 1$, $b = 2$, and $f'(x) = \frac{1}{x}$, the integral that gives the length of the curve $y = \ln x$ on the specified interval is $\int_1^2 \sqrt{1+\frac{1}{x^2}}\,dx$.

Answer to Exercise 4 (on page 7)

Since we are told the vector-valued velocity of the ship, we know that $\frac{dx}{dt} = -1412\sin t$, $\frac{dy}{dt} = 1412\cos t$, and $\frac{dz}{dt} = t$. The distance traveled in the first 60 seconds is given by $\int_0^{60} \sqrt{(-1412\sin t)^2 + (1412\cos t)^2 + t^2}\,dt$. Using a calculator, the integral evaluates to 84745 meters. The distance traveled in the second 60 seconds is given by
$\int_{60}^{120} \sqrt{(-1412\sin t)^2 + (1412\cos t)^2 + t^2}\,dt$. Using a calculator, this integral evaluates to 84898 meters.